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HOMOGENIZATION OF ELASTO-PLASTIC MATERIALS BY THE BOUNDARY ELEMENT METHOD

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Abstract

The formulation of the initial stress approach of the boundary element method (BEM) for two-dimensional elasto-plastic plates subjected to static tractions is presented. The developed computer code is used to analyze elasto-plastic materials with linear isotropic hardening which satisfy the Huber-Hencky-von Mises yield criterion. Representative volume elements (RVE) containing voids and inclusions are subjected to various boundary conditions. The relation between average stresses and average strains is computed for different volume fraction of voids and inclusions. The results are compared with the solutions computed by the finite element method (FEM). Effective yield stresses and tangent moduli are computed for different materials.

Key words: elasto-plastic material, boundary element method, homogenization, representative volume element, void, inclusion

1. INTRODUCTION

The aim of the work is an analysis of elasto-plastic materials containing voids and inclusions. Such materials can be efficiently analyzed by analytical (Dvorak, 2013) or computational methods. Among most popular computer methods of modelling porous and composite materials are the finite element method (FEM) and the boundary element method (BEM). Effective material properties can be computed numerically by considering a representative volume element (RVE). In order to obtain overall properties many RVE having different voids and inclusions should be considered. The main advantages of the BEM are: high accuracy for materials in complex stress state and easy modification of geometry. These properties of the method are particularly important in the analysis of RVE. The effective properties are required during the design of structures.

Ghosh and Moorthy (1995) developed a Voronoi Cell finite element method (VCFEM) to analyze small deformation of elasto-plastic arbitrary heterogeneous two-dimensional microstructures. Numerical results were compared with analytical solutions. The influence of shape, size, orientation and distribution of inclusions on micro- and macroscopic responses was investigated. Lee and Ghosh (1999) proposed two-scale analysis using the asymptotic homogenization method and the Voronoi cell finite element method for analysis of microstructures of porous and composite materials. The orthotropic elasticity tensor was obtained by analyzing microstructural problem with periodic boundary conditions. Parameters which characterize plastic behavior of the material were determined from microstructural RVE analyses with asymptotic homogeniza-

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tion. The results of the macroscopic analysis were compared with two-scale analysis with homogenization. Galvanetto et al. (1998) derived a homogenized constitutive relation for elasto-plastic periodic composites. The relation was obtained by analyzing unit cells subjected to large number of different boundary conditions. The method was proposed for two-dimensional unit cells subjected to monotonic proportional loading and small strains of the material. Kouznetsova et al. (2001) presented the micro-macro strategy for modeling non-linear heterogeneous materials at large deformations. The properties of the method were demonstrated for the aluminum plate subjected to the pure bending. The influence of the spatial distribution of heterogeneities on the overall macroscopic behavior was studied. Pierard et al. (2007) analyzed the uniaxial tension of composite containing aligned ellipsoidal inclusions embedded in an elasto-plastic matrix. The effective properties were obtained by the finite element analysis of an RVE. The results were used to assess the accuracy of a homogenization method. The non-linear behavior was modeled by the use of tangent and secant stiffness tensor of the phases. The effective stiffness tensor was computed by the Mori-Tanaka method. Brassart et al. (2010) presented micromechanical modeling of composites made of an elasto-plastic matrix with linear elastic spherical or ellipsoidal inclusions subjected to non-monotonic loading. The mean-field homogenization model was coupled with a finite element solution of the equivalent inclusion problem. The proposed approach was applied for the Mori-Tanaka method and dilute inclusion models.

The boundary element method has not been applied to analysis of elasto-plastic microstructures. In the present work materials containing voids and inclusions are analyzed by the initial stress approach of the boundary element method. Effective properties of materials are computed for different volume fractions of heterogeneities.

2. GOVERNING EQUATIONS

Consider an elasto-plastic, isotropic and homogenous body with the boundary Γ occupying the domain Ω (figure 1) which is subjected to tractions. The theory of small displacements is assumed.

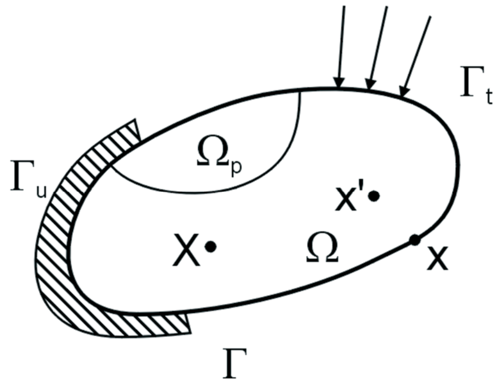

Fig. 1. *Elasto-plastic body*

The constitutive equations are based on the incremental stress-strain relations for inviscid plasticity and elasto-plastic flow theory (Telles, 1983):

$$d\sigma_{ij} = C_{ijkl}^{ep} d\varepsilon_{kl}\,, \tag{1}$$

where $d\varepsilon_{kl}$ is the total strain increment.

For the isotropic hardening:

$$C_{ijkl}^{ep} = C_{ijkl} - \frac{1}{\gamma'} C_{ijmn} a_{mn} a_{op} C_{opkl}\,, \tag{2}$$

where C_{ijkl} is the elastic tensor:

$$C_{ijkl} = \frac{2G\nu}{1-2\nu}\delta_{ij}\delta_{kl} + G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{3}$$

where G is the Kirchhoff shear modulus of elasticity, ν is the Poisson ratio, δ_{ij} is the Kronecker delta and a_{mn} is a derivative of the yield function F with respect to the appropriate component of stress tensor:

$$a_{mn} = \frac{\partial F}{\partial \sigma_{mn}}\,. \tag{4}$$

The parameter γ' also depends on derivatives of the yield function, the elastic tensor and the hardening slope H':

$$\gamma' = a_{ij} C_{ijkl} a_{kl} + H'\,. \tag{5}$$

Introducing the elastic stress increment:

$$d\sigma_{ij}^{e} = C_{ijkl} d\varepsilon_{kl}\,, \tag{6}$$

the plastic stress increment can be calculated as:

$$d\sigma_{ij}^{p} = d\sigma_{ij}^{e} - d\sigma_{ij} = \frac{1}{\gamma'} C_{ijmn} a_{mn} a_{kl} d\sigma_{kl}^{e}\,. \tag{7}$$

3. BOUNDARY INTEGRAL EQUATIONS

The fundamentals of the BEM formulation for nonlinear materials are presented in the textbooks by Telles (1983), Banerjee (1994) and Gao and Davies (2002). The most popular approaches in the elasto-plastic BEM are the initial stress and initial strain formulation, because of easy physical interpretation and simple implementation. Each approach requires the plastic domain discretization, but this procedure does not increase the number of degrees of freedom. The integral equations are formulated in the incremental form and the numerical solution is obtained by the iterative methods.

The relation between the mechanical fields can be obtained using the displacement integral equation. For the initial stress approach, the equation has the form (Telles, 1983):

$$c_{ij}(x')u_j(x') = \int_\Gamma U_{ij}(x',x)t_j(x)d\Gamma(x) - \int_\Gamma T_{ij}(x',x)u_j(x)d\Gamma(x) + \int_\Omega E_{jki}(x',X)\sigma^p_{jk}(X)d\Omega(X), \tag{8}$$

where c_{ij} is a constant, which depends on the position of the collocation point, u_j and t_j are components of displacements and tractions, respectively, U_{ij}, T_{ij}, E_{jki} are fundamental solutions of elastostatics, x' is the collocation point, x is a boundary point and X is a domain point (figure 1).

Contrary to the elastostatic case, equation (8) contains the domain plastic term, which depends on the unknown plastic stress σ^p_{jk}. In order to obtain the stress fields in the domain the stress integral equation is used. For the initial stress approach the equation is (Telles, 1983):

$$\sigma_{ij}(x') = \int_\Gamma U_{ijk}(x',x)t_k(x)d\Gamma(x) - \int_\Gamma T_{ijk}(x',x)u_k(x)d\Gamma(x) + \int_\Omega E_{ijkl}(x',X)\sigma^p_{kl}(X)d\Omega(X) + F_{ijkl}\sigma^p_{kl}(x') \tag{9}$$

where U_{ijk}, T_{ijk}, E_{ijkl} and F_{ijkl} are other fundamental solutions.

4. NUMERICAL IMPLEMENTATION

In order to obtain the numerical solution, the boundary is divided into 3-node boundary elements and the part of the body where the inelastic behaviour is expected is discretized into 8-node quadrilateral cells, as shown in figure 2 (Czyż, Fedeliński, 2006). The domain discretization is consistent with the boundary discretization i.e. one quadratic cell adjoins one quadratic boundary element (figure 2). The method requires discretization of this part of the domain Ω_p, which is in the plastic state. The boundary coordinates, displacements and tractions are interpolated using quadratic shape functions and the stresses in the domain using quadratic shape functions.

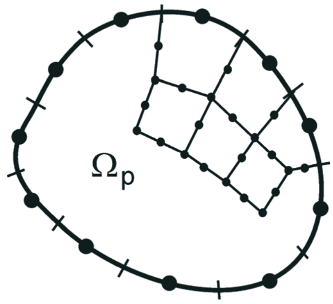

Fig. 2. *Discretization of the body by using quadratic boundary elements and cells*

The displacement integral equation (8) is applied to every boundary node. The resulting system of equations can be written in the matrix form as:

$$\boldsymbol{Hu} = \boldsymbol{Gt} + \boldsymbol{E\sigma}^p, \tag{10}$$

where $\boldsymbol{H}$ and $\boldsymbol{G}$ depend on boundary integrals of the fundamental solutions T_{ij} and U_{ij}, respectively, and boundary shape functions, $\boldsymbol{E}$ is dependent on the fundamental solution E_{jki} and domain shape functions, $\boldsymbol{u}$ and $\boldsymbol{t}$ contain boundary nodal displacements and tractions, respectively and $\boldsymbol{\sigma}^p$ contains components of plastic stress tensor.

The stress integral equations (9) are used to determine stresses at all internal nodes of cells. These equations can be written in the matrix form as:

$$\boldsymbol{\sigma} = \boldsymbol{G}'\boldsymbol{t} - \boldsymbol{H}'\boldsymbol{u} + \boldsymbol{E}'\boldsymbol{\sigma}^p, \tag{11}$$

where $\boldsymbol{\sigma}$ contains components of the stress tensor at all internal nodes of cells. The matrices ***G'***, ***H'*** and ***E'*** are obtained similarly as matrices in equation (10). They depend on appropriate fundamental solutions in equation (9). The stresses in nodes of the cells at the boundary are computed using tractions and displacements.

The materials with inclusions are analyzed by the subregion method (Banerjee, 1994). Continuity of displacements and equilibrium of tractions along interfaces are assumed.

5. SOLUTION PROCEDURE

The displacement equation and stress equation contains a vector of not known a priori inelastic stresses $\boldsymbol{\sigma}^p$. In order to determine the stresses the tractions are gradually increased and an iterative procedure is used for each load level. At the beginning of the iterative procedure the plastic stresses are zero. Next the boundary unknowns are computed using equation (10) and the elastic stresses using equation (11). Later the yield criterion is applied to detect which nodes are in the plastic state. For each node, where the effective stresses are above the yield stress, the elastic stress increment is computed. The elastic stress increment is the excess of stresses over the stresses, which correspond to the yield stress. Knowing the elastic stress increment, the plastic stress increment can be computed from equation (7) and substituted again to equation (10) for each node in the plastic state. Next this increment is added to the total plastic stresses in the node. When the plastic stress increment is determined for each node, the boundary unknowns with the new vector of plastic stresses are computed. This procedure is repeated until the change of the results is so small that can be neglected.

6. EXAMPLES

A square representative volume element (RVE) shown in figure 3 contains 9 circular voids or inclusions. The length of the edges of the element is l=3 mm, the distance between centres of the voids/inclusions is a=1 mm and their diameter is d (figure 3a). Three different diameters of voids/inclusions are considered d=0.3; 0.5; 0.7 mm. The elasto-plastic material with linear isotropic hardening satisfies the Huber-Hencky-von Mises yield criterion. The RVE is subjected to tensile q_1 or shear loading q_2, as shown in figure 3b and 3c, respectively. The displacements of nodes on the external boundary of the RVE are used to compute average strains. The RVE is analyzed using the BEM and the FEM. The external boundaries of the RVE are divided into 60 elements and each void/inclusion into 10 boundary elements.

6.1. Representative volume element with voids

The material properties of the plate are: the Young modulus E = 200000 MPa, the tangent modulus H = 50000 MPa, the yield stress σ_Y = 300 MPa and the Poisson ratio ν = 0.3. The plate is in plane stress state. The relation between applied stress and average strain for tensile and shear loading is shown in figure 4a and figure 4b, respectively. The results computed by the BEM and the FEM agree very well.

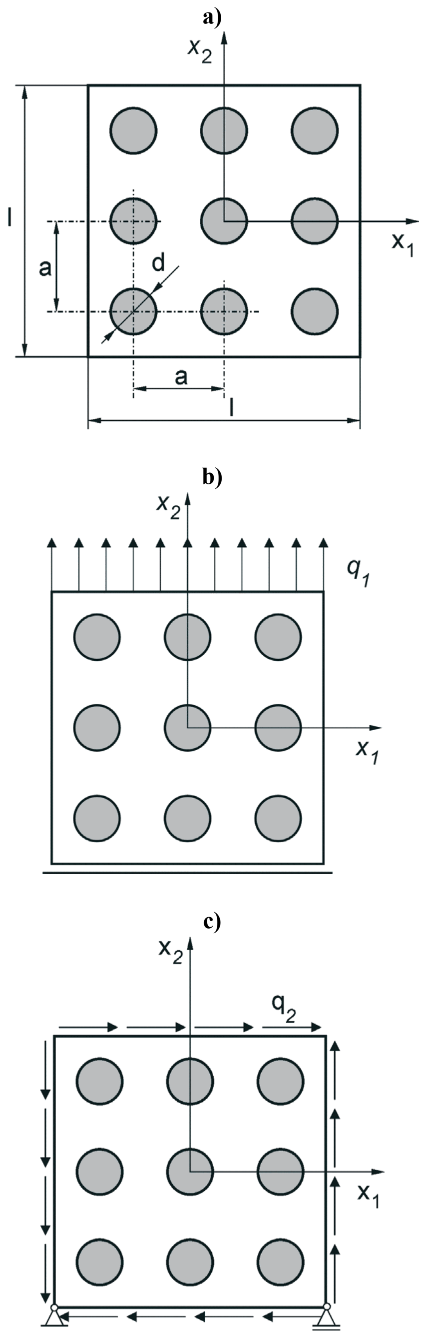

Fig. 3. *Representative volume element: a) dimensions, b) tensile loading, c) shear loading*

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The effective plastic modulus and the tangent modulus are smaller than the appropriate quantities for the homogeneous material. The RVE with voids subjected to shear loading is less stiffer than the same RVE subjected to tensile loading in comparison with the homogeneous material.

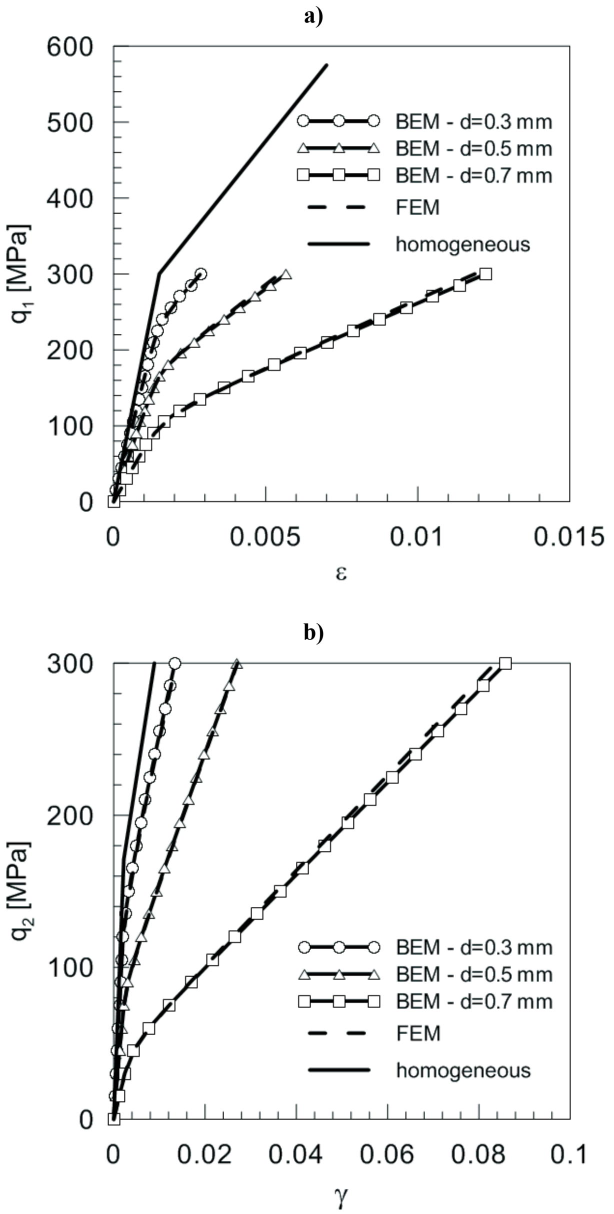


Fig. 4. Applied stress vs. average strain for the RVE with voids: a) tensile loading q_1, b) shear loading q_2

6.2. Representative volume element with inclusions

The material properties of the matrix are: the Young modulus E = 100000 MPa, the tangent modulus H = 25000 MPa, the yield stress σ_Y = 150 MPa and the Poisson ratio ν = 0.3. The material properties of the inclusion are: the Young modulus E = 200000 MPa, the tangent modulus H = 50000 MPa, the yield stress σ_Y = 300 MPa and the Poisson ratio ν = 0.3. The plate is in plane stress state. The relation between applied stress and average strain for tensile and shear loading is shown in figure 5a and 5b, respectively. The results computed by the BEM and the FEM agree very well. The stress distributions in the RVE with inclusions (d = 0.7 mm) subjected to tensile and shear final loading 300 MPa are shown in figure 6. The highest normal stresses σ_{22} for tension occur along vertical lines crossing inclusions. The highest shear stresses σ_{12} occur at points which are at the same distance from the neighbouring inclusions.

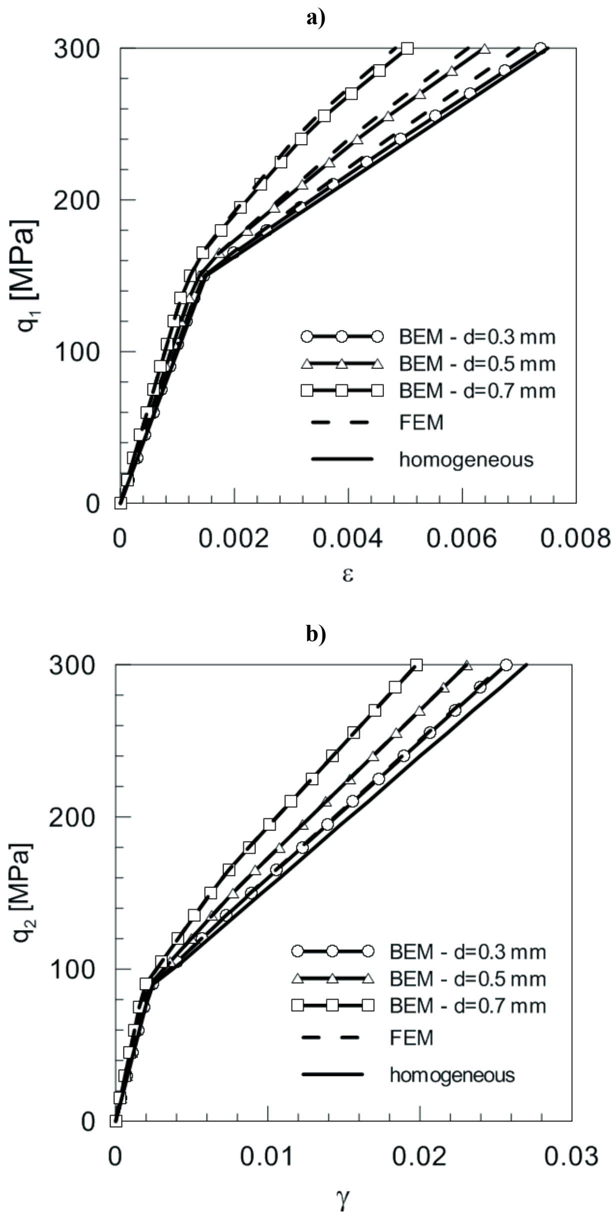
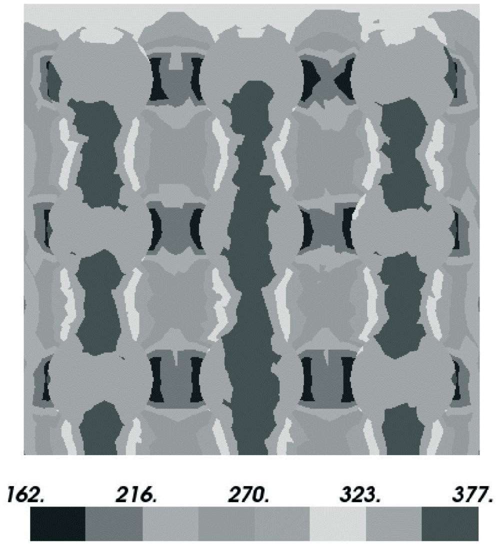


Fig. 5. Applied stress vs. average strain for the RVE with inclusions: a) tensile loading q_1, b) shear loading q_2

a)



b)

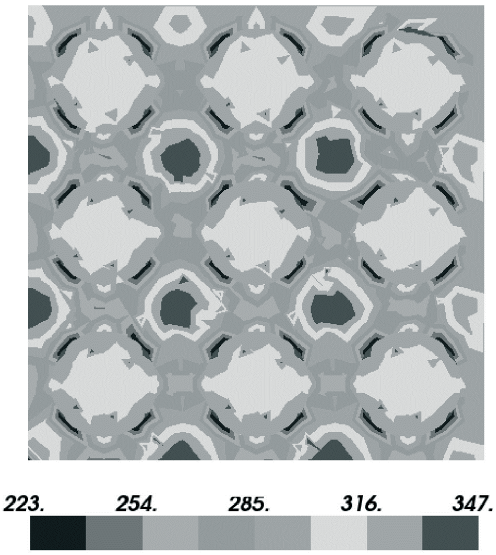


Fig. 6. Stress distribution in the RVE with inclusions: a) tensile loading – normal stress σ_{22} [MPa], b) shear loading – shear stress σ_{12} [MPa]

The stress distributions shown in figure 6 are not perfectly symmetric because the meshes of internal cells are not regular.

6.3. Effective properties of materials

The same constitutive model for the homogenized material is assumed as for the material at the micro scale. The linear parts of the plots are approximated for the material in the elastic and plastic state and used to compute the effective yield stresses and the effective tangent moduli. The effective yield stress is the stress, which corresponds to the point of intersection of straight lines for the material in the elastic and plastic state. The effective properties for the material with voids and inclusions subjected to tensile and shear loading are shown in table 1 and 2.

***Table 1.** Effective yield stress of the material [MPa]*

diameter d [mm]	void		inclusion	
	tension	shear	tension	shear
0.0	300	171	150	85
0.3	236	149	154	90
0.5	175	95	168	96
0.7	117	55	186	108

***Table 2.** Effective tangent modulus of the material [MPa]*

diameter d [mm]	void		inclusion	
	tension	shear	tension	shear
0.0	50000	19200	25000	8600
0.3	44500	13800	24600	8900
0.5	29500	8400	26900	9700
0.7	17200	3000	31800	10900

In order to analyse convergence and accuracy of the method, the RVE containing voids/inclusions of diameter d = 0.5 mm subjected to tension are considered using two different meshes of boundary elements and internal cells. The coarse mesh, as in the above numerical examples, contains 60 boundary elements for the external boundary of the RVE and 10 boundary elements for each void/inclusion (the total number of elements is 150). The matrix is discretized using 308 and each inclusion using 16 internal cells. The dense mesh have 96 boundary elements for the external boundary and 16 boundary elements for each void/inclusion (the total number of elements is 240). The matrix is discretized using 386 and each inclusion using 32 internal cells. The relative difference of average strains for the two types of discretization is computed using the following norm:

$$\delta = \frac{\sqrt{\sum_{i=1}^{n} (\varepsilon_2 - \varepsilon_1)^2}}{n\varepsilon_{2\max}}, \qquad (12)$$

where: n is the number of load levels, ε_1 is the average strain for the coarse mesh, ε_2 is the average strain for the dense mesh and ε_{2max} is the maximal average strain for the fine mesh during the loading process.

The computed average differences of strains are: for the material with voids $\delta = 0.061\%$ and for the material with inclusion $\delta = 0.076\%$.

The very small differences between the results show that the convergence is achieved for the coarse meshes.

7. CONCLUSIONS

The relation between average stresses and strains in elasto-plastic representative volume elements containing voids and inclusions subjected to tensile and shear loadings are computed using the boundary element method and the finite element method. The results computed by two methods agree very well. The numerical results are used to compute effective yield stresses and tangent moduli for different volume fraction of voids or inclusions. The influence of volume fraction of voids and inclusions on overall material properties is studied.

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HOMOGENIZACJA MATERIAŁÓW SPRĘŻYSTO-PLASTYCZNYCH ZA POMOCĄ METODY ELEMENTÓW BRZEGOWYCH

Streszczenie

W pracy przedstawiono sformułowanie naprężeń początkowych metody elementów brzegowych (MEB) w analizie dwuwymiarowych tarcz obciążonych statycznie siłami powierzchniowymi. Opracowany program komputerowy zastosowano do analizy materiałów sprężysto-plastycznych z liniowym wzmocnieniem izotropowym, który spełnia warunek plastyczności Hubera-Hencky'ego-von Misesa. Analizowano reprezentatywne elementy objętościowe (RVE) poddane różnym warunkom brzegowym zawierające pustki i wtrącenia. Określono związki między średnimi odkształceniami i naprężeniami dla różnego udziału objętościowego pustek i wtrąceń oraz różnych warunków brzegowych. Wyniki porównano z rozwiązaniami otrzymanymi metodą elementów skończonych (MES). Wyznaczono zastępcze granice plastyczności i moduły styczne dla różnych materiałów.

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