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IDENTIFICATION OF INTERFACE POSITION IN TWO-LAYERED DOMAIN USING GRADIENT METHOD COUPLED WITH THE BEM

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Abstract

The non-homogeneous domain being the composition of two sub-domains is considered, at the same time the position of internal interface is unknown. The additional information necessary to solve the identification problem results from the knowledge of temperature field at the set of points X selected from the domain analyzed. From the practical point of view the points X should be located at the external surface of the system. The steady temperature field in domain considered is described by two energy equations (the Laplace equations), continuity condition given on the contact surface and the boundary conditions given on the external surface of domain. To solve the inverse problem the gradient method is used. The sensitivity coefficients appearing in the final form of equation which allows one to find the solution using a certain iterative procedure are determined by means of the implicit approach of shape sensitivity analysis. This approach is especially convenient in the case of boundary element method application (this method is used at the stage of numerical algorithm construction). In the final part of the paper the examples of computations are shown.

Key words: heat transfer, inverse problem, gradient method, boundary element method

1. INTRODUCTION

The following boundary value problem is considered

$$(x,y)\in\Omega_e: \quad \lambda_e\left[\frac{\partial^2 T_e(x,y)}{\partial x^2}+\frac{\partial^2 T_e(x,y)}{\partial y^2}\right]=0, \quad e=1,2 \tag{1}$$

where index e corresponds to the respective sub-domains, λ_e is the thermal conductivity, T, x, y denote the temperature and spatial co-ordinates, respectively. The equation (1) is supplemented by the typical boundary conditions, in particular

$$(x,y)\in\Gamma_{ex}: \quad -\lambda_1\frac{\partial T_1(x,y)}{\partial n}=\alpha\left[T_1(x,y)-T_a\right] \tag{2}$$

where Γ_{ex} is the external surface of domain marked in figure 1, T_a is the ambient temperature, α is the heat transfer coefficient, $\partial T_1/\partial n$ denotes the normal derivative.

On the surface between sub-domains the continuity of heat flux and temperature field is assumed, this means

$$(x,y)\in\Gamma_c: \quad \begin{cases} -\lambda_1\dfrac{\partial T_1(x,y)}{\partial n}=\lambda_2\dfrac{\partial T_2(x,y)}{\partial n} \\ T_1(x,y)=T_2(x,y) \end{cases} \tag{3}$$

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On the internal surface Γ_{in} (c.f. figure 1) the Dirichlet condition is taken into account

$$(x,y)\in\Gamma_{in}: \quad T_2(x,y)=T_b \tag{4}$$

On the remaining parts of boundary the no-flux condition can be accepted.

As is well known, when the thermophysical and geometrical parameters appearing in the mathematical model of the process considered are given then the direct problem is formulated and the temperature distribution in the domain Ω can be found.

The inverse problem considered here bases on the assumption that the temperature distribution at the boundary Γ_{ex} is known (e.g. thermographs), while the position of Γ_c is unknown (Ciesielski & Mochnacki, 2012; Romero Mendez et al., 2010).

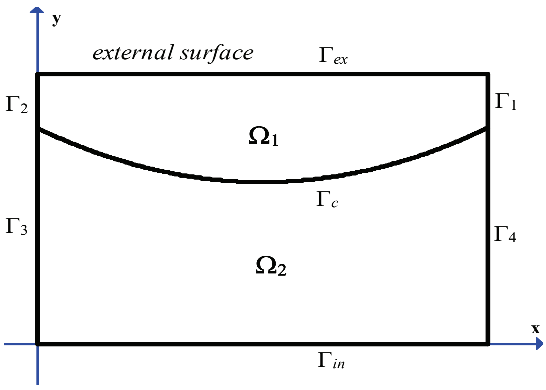

Fig. 1. *Domain considered*

2. SOLUTION OF DIRECT PROBLEM BY MEANS OF THE BOUNDARY ELEMENT METHOD

The boundary integral equations corresponding to the Laplace equations (1) are the following (Brebbia & Dominguez, 1992 ; Majchrzak, 2001)

$$(\xi,\eta)\in\Gamma_e: \quad B(\xi,\eta)T_e(\xi,\eta)+ \int_{\Gamma_e} T_e^*(\xi,\eta,x,y)\; q_e(x,y)\mathrm{d}\Gamma_e = \int_{\Gamma_e} q_e^*(\xi,\eta,x,y)\; T_e(x,y)\mathrm{d}\Gamma_e \tag{5}$$

where $B(\xi, \eta)\in(0, 1)$ is the coefficient connected with the local shape of boundary, (ξ, η) is the observation point, $q_e(x, y) = -\lambda_e \partial T_e(x, y)/\partial n$, $n = [n_x, n_y]$ and $T_e^*(\xi, \eta, x, y)$ is the fundamental solution

$$T_e^*(\xi,\eta,x,y)=\frac{1}{2\pi\lambda_e}\ln\frac{1}{r} \tag{6}$$

where r is the distance between the points (ξ, η), (x, y) and

$$q_e^*(\xi,\eta,x,y)=-\lambda_e\frac{\partial T_e^*(\xi,\eta,x,y)}{\partial n}=\frac{d}{2\pi r^2} \tag{7}$$

while

$$d=(x-\xi)n_x+(y-\eta)n_y \tag{8}$$

In numerical realization of the BEM the boundaries are divided into boundary elements and the integrals appearing in equations (5) are substituted by the sums of integrals over these elements. After the mathematical manipulations one obtains two systems of algebraic equations (Majchrzak, 2001)

$$\mathbf{G}^e\mathbf{q}^e=\mathbf{H}^e\mathbf{T}^e \tag{9}$$

Now, the following notation is introduced (c.f. figure 1)

– $\mathbf{T}_1^1$, $\mathbf{T}_1^2$, $\mathbf{T}_1^{ex}$, $\mathbf{q}_1^1$, $\mathbf{q}_1^2$, $\mathbf{q}_1^{ex}$ are the vectors of functions T and q at the boundary $\Gamma_1\cup\Gamma_2\cup\Gamma_{ex}$ of domain Ω_1,
– $\mathbf{T}_{c1}$, $\mathbf{T}_{c2}$, $\mathbf{q}_{c1}$, $\mathbf{q}_{c2}$ are the vectors of functions T and q on the contact surface Γ_c between sub-domains Ω_1 and Ω_2,
– $\mathbf{T}_2^3$, $\mathbf{T}_2^4$, $\mathbf{T}_2^{in}$, $\mathbf{q}_2^3$, $\mathbf{q}_2^4$, $\mathbf{q}_2^{in}$ are the vectors of functions T and q at the boundary $\Gamma_3\cup\Gamma_4\cup\Gamma_{in}$ of domain Ω_2

and then one has

– for sub-domain Ω_1

$$\begin{bmatrix}\mathbf{G}_1^1 & \mathbf{G}_1^{ex} & \mathbf{G}_1^2 & \mathbf{G}_{c1}\end{bmatrix}\begin{bmatrix}\mathbf{q}_1^1\\ \mathbf{q}_1^{ex}\\ \mathbf{q}_1^2\\ \mathbf{q}_{c1}\end{bmatrix}=\begin{bmatrix}\mathbf{H}_1^1 & \mathbf{H}_1^{ex} & \mathbf{H}_1^2 & \mathbf{H}_{c1}\end{bmatrix}\begin{bmatrix}\mathbf{T}_1^1\\ \mathbf{T}_1^{ex}\\ \mathbf{T}_1^2\\ \mathbf{T}_{c1}\end{bmatrix} \tag{10}$$

– for sub-domain Ω_2

$$\begin{bmatrix}\mathbf{G}_{c2} & \mathbf{G}_2^3 & \mathbf{G}_2^{in} & \mathbf{G}_2^4\end{bmatrix}\begin{bmatrix}\mathbf{q}_{c2}\\ \mathbf{q}_2^3\\ \mathbf{q}_2^{in}\\ \mathbf{q}_2^4\end{bmatrix}=\begin{bmatrix}\mathbf{H}_{c2} & \mathbf{H}_2^3 & \mathbf{H}_2^{in} & \mathbf{H}_2^4\end{bmatrix}\begin{bmatrix}\mathbf{T}_{c2}\\ \mathbf{T}_2^3\\ \mathbf{T}_2^{in}\\ \mathbf{T}_2^4\end{bmatrix} \tag{11}$$

The continuity condition (3) written in the form

$$\begin{cases}\mathbf{q}_{c1}=-\mathbf{q}_{c2}=\mathbf{q}\\ \mathbf{T}_{c1}=\mathbf{T}_{c2}=\mathbf{T}\end{cases} \tag{12}$$

allows one to couple the equations (10), (11). Taking into account the remaining boundary conditions finally one obtains

$$\begin{bmatrix} -\mathbf{H}_1^1 & \alpha\mathbf{G}_1^{ex}-\mathbf{H}_1^{ex} & -\mathbf{H}_1^2 & -\mathbf{H}_{c1} & \mathbf{G}_{c1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{H}_{c2} & -\mathbf{G}_{c2} & -\mathbf{H}_2^3 & \mathbf{G}_2^{in} & -\mathbf{H}_2^4 \end{bmatrix} \begin{bmatrix} \mathbf{T}_1^1 \\ \mathbf{T}_1^{ex} \\ \mathbf{T}_1^2 \\ \mathbf{T} \\ \mathbf{q} \\ \mathbf{T}_2^3 \\ \mathbf{q}_2^{in} \\ \mathbf{T}_2^4 \end{bmatrix} = \begin{bmatrix} \alpha\mathbf{G}_1^{ex}T_a \\ \mathbf{G}_2^{in}T_b \end{bmatrix} \tag{13}$$

or

$$\mathbf{AY}=\mathbf{B} \tag{14}$$

where **A** is the main matrix of the system of equations (13), **Y** is the vector of unknowns and **B** is the free terms vector.

The system of equations (14) allows one to find the ,missing'' boundary values. Knowledge of nodal boundary temperatures and heat fluxes constitutes a basis for computations of internal temperatures at the optional set of points selected from the domain considered.

3. SOLUTION OF INVERSE PROBLEM USING GRADIENT METHOD COUPLED WITH THE BEM

The inverse problem considered here bases on the assumption that the temperature distribution at the boundary Γ_{ex} is known, while the position of Γ_c is unknown. The surface Γ_c is defined by the set of points (x_n, y_n), $n = 1, 2, \ldots, N$. The aim of investigations is to determine the values of shape parameters $b_1, b_2, \ldots, b_N$ which correspond to the co-ordinates y_n shown in figure 2.

The criterion which should be minimized is of the form (Kurpisz & Nowak, 1995; Burczyński, 2003)

$$S(b_1,\ldots,b_n,\ldots,b_N)=\frac{1}{M}\sum_{i=1}^{M}(T_i-T_{di})^2 \tag{15}$$

where T_{di} , T_i are the temperatures known from the measurements and calculated ones, respectively. In this paper the real measurements are substituted by the temperatures T_i obtained from the direct problem solution for arbitrary assumed position of points (x_n, y_n).

Using the necessary condition of optimum, one obtains

$$\frac{\partial S}{\partial b_n}=\frac{2}{M}\sum_{i=1}^{M}(T_i-T_{di})\frac{\partial T_i}{\partial b_n}=0, \qquad n=1,2,\ldots,N \tag{16}$$

The function T_i is expanded into the Taylor series taking into account the first derivatives

$$T_i=T_i^k+\sum_{j=1}^{N}\left(\frac{\partial T_i}{\partial b_j}\right)_{b_j=b_j^k}(b_j^{k+1}-b_j^k) \tag{17}$$

where b_j^0 is the arbitrary assumed value of parameter b_j, while for $k > 0$ it results from previous iteration.

Introducing (17) into (16) one has

$$\sum_{i=1}^{M}\sum_{j=1}^{N}U_{i,j}^kU_{i,n}^k(b_j^{k+1}-b_j^k)=$$
$$\sum_{i=1}^{M}\left(T_{di}-T_i^k\right)U_{i,n}^k \qquad n=1,2,\ldots,N \tag{18}$$

where

$$U_{i,j}^k=\left(\frac{\partial T_i}{\partial b_j}\right)_{b_j=b_j^k} \tag{19}$$

are the sensitivity coefficients. From the system of equations (18) the values b_j^{k+1} are calculated. To determine the sensitivity coefficients the methods of shape sensitivity analysis are used (Kleiber, 1997; Majchrzak et al., 2011; Freus et al., 2012). Here the implicit differentiation method, which belongs to the discretized approach, basing on the differentiation of algebraic boundary element matrix equations (14) is applied. So, the system of equations (14) should be differentiated with respect to parameter b_j and then

$$\frac{\partial\mathbf{A}}{\partial b_j}\mathbf{Y}+\mathbf{A}\frac{\partial\mathbf{Y}}{\partial b_j}=\frac{\partial\mathbf{B}}{\partial b_j} \tag{20}$$

or

$$\mathbf{A}\frac{\partial \mathbf{Y}}{\partial b_j} = \frac{\partial \mathbf{B}}{\partial b_j} - \frac{\partial \mathbf{A}}{\partial b_j}\mathbf{Y} \qquad (21)$$

It should be pointed out that the derivatives of the boundary element matrices are calculated analytically (Majchrzak et al., 2011).

4. RESULTS OF COMPUTATIONS

The domain of dimensions $2L \times L$ (L = 0.02 m) shown in figure 1 has been considered. At first, the direct problem described in the chapter 1 has been solved. The following input data have been introduced: thermal conductivities λ_1 = 0.1 W/(mK), λ_2 = 0.2 W/(mK), heat transfer coefficient α = 10 W/(m^{2}K), ambient temperature T_a = 20°C (c.f. condition (2)), boundary temperature T_b = 37°C (c.f. condition (4)). The shape of internal surface Γ_c has been assumed in the form of parabolic function (in this place the optional shapes can be taken into account)

$$y(x) = \frac{0.8L(x-L)^2 - y_p x(x-2L)}{L^2}, \qquad (22)$$

where (L, y_p) = (0.02 m, 0.012 m) is the tip of parabola. The discretization of boundaries using the linear boundary elements is shown in figure 2.

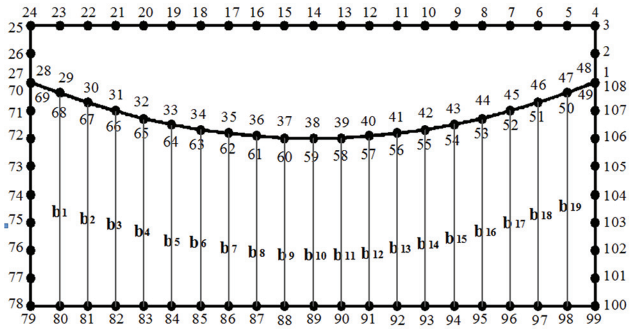

Fig. 2. Discretization of boundaries

In figure 3 the temperature distribution in the domain considered is presented, while figure 4 illustrates the course of temperature at the external surface.

To solve the inverse problem, 29 shape sensitivity coefficients corresponding to the y co-ordinate of nodes from 29 = 68 to 47 = 50 (figure 2) has been distinguished. The nodes 28 and 48 are fixed – c.f. equation (20). So, 29 additional problems connected with the determination of sensitivity functions have been formulated.

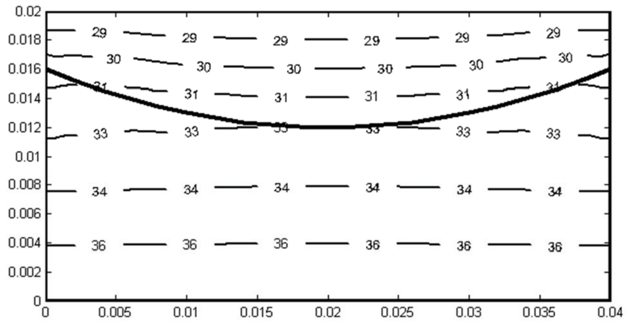

Fig. 3. Temperature distribution in the domain considered

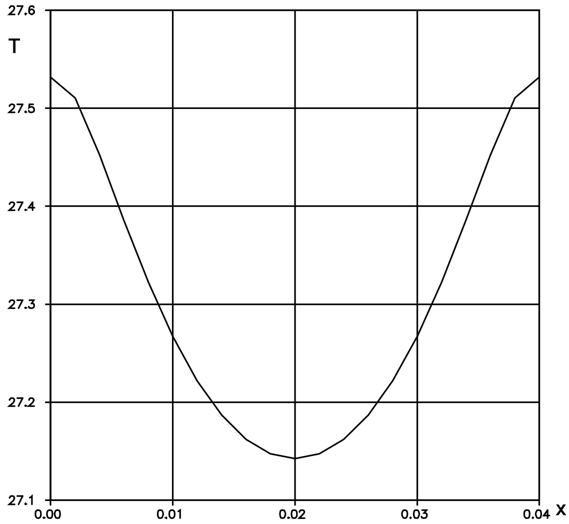

Fig. 4. Temperature distribution at the external surface

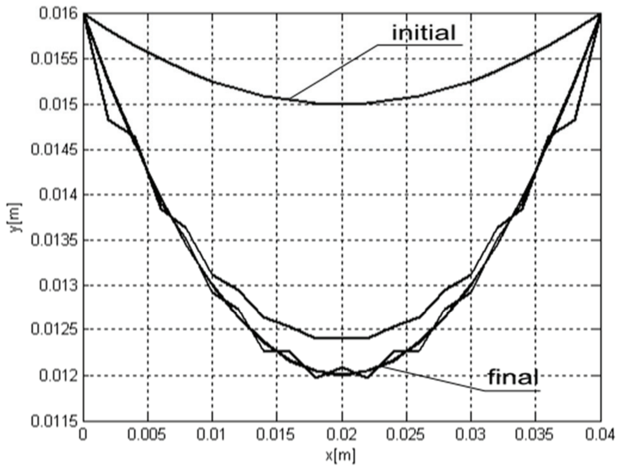

Fig. 5. Results of identification – variant 1

The identification problem has been solved under the assumption that the temperatures at the nodes from 5 to 23 (figure 2) are known and the initial position of internal boundary is described by function (22) where y_p = 0.015 m or y_p = 0.011 m (the different start points allow ones to observe the course of iteration process). In figures 5 and 6 the results of computations are shown. It is visible that for exact input data the exact position of boundary is obtained and the iteration process is convergent.

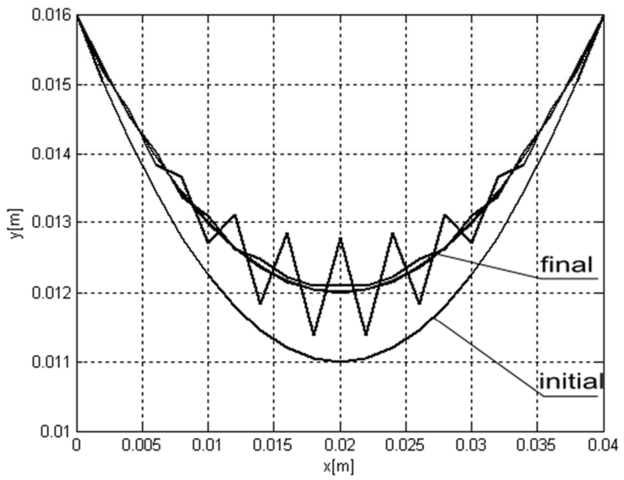

Fig. 6. *Results of identification – variant 2*

6. CONCLUSIONS

The algorithm proposed can be useful, among others, in the medical practice (estimation of wound shape on the basis of surface temperature distribution). It should be pointed out that both from the mathematical and numerical points of view the problem is rather complicated, but taking into account the practical applications it seems that the scientific research in this scope should be continued. The algorithm proposed allows one to identify the complex shapes of internal boundary (the co-ordinates y_n are estimated separately). In a similar way the 3D problems can be also solved. In the future the detailed research of iterative procedure convergence should be also done.

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WYKORZYSTANIE METODY GRADIENTOWEJ I MEB DO IDENTYFIKACJI KSZTAŁTU GRANICY MIĘDZY PODOBSZARAMI W DWUWARSTWOWYM NIEJEDNORODNYM OBSZARZE CIAŁA STAŁEGO

Streszczenie

W pracy rozpatruje się niejednorodny obszar ciała stałego będący złożeniem dwóch podobszarów, przy czym położenie powierzchni granicznej nie jest znane. Dodatkową informacją pozwalającą rozwiązać sformułowane w ten sposób zadanie odwrotne są wartości temperatury w punktach X wyróżnionych w rozpatrywanym obszarze. Z praktycznego punktu widzenia punkty przyłożenia sensorów powinny być zlokalizowane na powierzchni zewnętrznej pozostającej w kontakcie z otoczeniem. Model matematyczny procesu tworzy układ równań eliptycznych (równań Laplace'a), warunki idealnego kontaktu na powierzchni kontaktu i warunki zadane na powierzchniach zewnętrznych. Rozwiązanie zadania uzyskano metodą gradientową, a współczynniki wrażliwości występujące w układzie rozwiązującym wyznaczono wykorzystując niejawne podejście analizy wrażliwości, które jest szczególnie efektywne w przypadku zastosowania metody elementów brzegowych (tę metodę wykorzystano na etapie konstrukcji algorytmu numerycznego). W końcowej części artykułu zamieszczono wyniki obliczeń numerycznych.

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