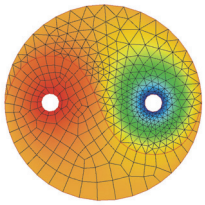
Publishing House
AKAPIT

COMPUTER METHODS IN MATERIALS SCIENCE

Informatyka w Technologii Materiałów

Vol. 10, 2010, No. 4



THEORETICAL AND NUMERICAL ISSUES ON DUCTILE FAILURE PREDICTION – AN OVERVIEW

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Abstract

The main goal of this paper is to give a general overview of some of the recent advances accomplished in the description of ductile damage, both from a theoretical and numerical point of view. To start with, the classical local theory with regard to the thermodynamics of irreversible processes is reviewed where a general elasto-plastic damage model is established. It is also highlighted the assumptions and limitations behind the classical theory when the constitutive equations are obtained from the solution of a constrained maximisation problem. Recent advances on the non-local modelling of ductile damage are also addressed where we shed some light on the principles and consequences of non-locality in elasto-plastic damage models. The issues regarding the efficient numerical implementation of both local and non-local theories are also discussed where special attention is devoted to the implementation of non-local models. In particular, a novel computational strategy, suitable for implementations in commercial programs, is presented for the explicit finite element code LS-DYNA in detail. A FORTRAN code excerpt is given in which the main steps for the implementation of the model are schematically depicted. The effectiveness of the non-local model is assessed through the simulation of an axisymmetric specimen and a sheet metal forming process. It is shown that in both cases the non-local numerical strategy is able to diminish the pathological mesh dependency inherently present in local elasto-plastic damage models.

Key words: ductile failure, damage, elasto-plasticity, non-local model, LS-DYNA user-defined material

1. INTRODUCTION

Metal forming processes are generally characterized by the fact that they involve significant changes in shape of a part, in the solid state, through the material flow under large plastic deformation, controlled by contact and friction by means of tools. Their industrial importance comes from the large variety of structural parts manufactured by these processes in a large diversity of industrial sectors (automotive, aeronautics, consumption goods, etc.). The state and geometry of the final part depend on various factors: load conditions, geometry of matrices, tools, lubrication of contact zones, geometry of pre-forms and material forming limits, to name only the most important ones. Therefore, the possibility of mastering all these factors is decisive for development and optimization of metal forming processes based on large plastic deformation of metals. In this aspect numerical tools based mainly on the Finite Element Method have become indispensable. The last twenty years have witnessed a great development and evolution on both theoretical and numerical abilities to deal with these phenomena. To a large extent it is possible today to predict deformation mechanisms, stresses, forces, material proper-

 ISSN 1641-8581

ties changes, the influence of tools geometry, lubrication conditions, etc. The great development in computing facilities made way for important theoretical advances on the treatment of large deformations of inelastic materials which could not be assessed before, which in turn prompted the development of new and faster algorithms to solve increasingly complex problems.

Nowadays there exist powerful commercial codes which are able to simulate complex forming processes and give very useful information for their improvement and optimization. However, one aspect that needs still some progress and in which commercial codes commonly fail to give the adequate response is the prediction of material formability, under complex loading paths. Nevertheless, this is a decisive feature in order to be able to predict defective parts in processes like forging or to describe processes in which fracture is a part of the process itself as in sheet blanking or metal cutting. In large deformation of metals, when plastic deformation reaches a threshold level, which may depend on the loading, the fatigue limit and the ultimate stress, a ductile damage process may occur concomitantly with the plastic deformation due to the nucleation, growth and coalescence of micro-voids. Sometimes those codes include *a posteriori* fracture indicators that are not always suitable for all deformation paths. In the simulation of bulk forming processes, it is common to utilise fracture criteria based on the computational evaluation of functions of some state variables and that depend on the deformation story. Those criteria may be, broadly, classified in two groups: one based on micromechanics which include as primer state variable the total plastic work (Freudenthal, 1950), the maximum plastic shear work or the equivalent plastic strain (Datsko, 1966); and another based on the growth of defects which include geometric aspects (McClintock, 1968; Rice & Tracey, 1969), growth mechanisms, dependent on principal stresses (Cockcroft & Latham, 1968) or hydrostatic pressure (Norris et al., 1978; Atkins, 1981), or material behaviour coupling (Oyane et al., 1978; Tai & Yang, 1987; Lemaitre, 1985).

In fact, in many cases those criteria do not take into account the facts that the damage localization site may be away from the zones where the maximum equivalent plastic deformation is located or that damage evolution may be different for compression or traction stress states or different triaxiality stress states. For these processes, models based on Continuous Damage Mechanics (CDM) (Lemaitre, 1996) may offer a better understanding of the physical phenomenon and become an important tool in the formability and fracture prediction. More recently, a lot of research is being focused on multi-scales models which may give an important insight on how the damage mechanisms at lower scales may manifest at the macro level and affect the definition of the phenomenological laws which are usually adopted at this level. Nevertheless, due to the huge amount of computer capabilities needed, in most cases nonexistent nowadays, the application of these multi-scale models in practical terms are still far from being a reality and for the moment industry will have to rely on the application of phenomenological models at the macro scale.

The softening induced by the standard implementation of those damage models in finite element solutions leads to mesh and orientation dependence as the localization effects are not correctly dealt with by mesh refinement. One of the solutions for this problem is the use non-local models (Pijaudier-Cabot & Bažant, 1987; De Borst & Mühlhaus, 1992; De Vree et al., 1995; Strömberg & Ristinmaa, 1996; Polizzotto et al., 1998; Borino et al., 1999; Jirásek & Rolshoven, 2003; Cesar de Sa et al., 2006; Cesar de Sa & Zheng, 2007; Jirásek, 2007; Andrade et al., 2009a). Non-local models include some length scale information, related with localization effects due to microstructure heterogeneity, in order to average an internal variable associated with the dissipative process. Two types of models are usually assumed for this purpose: integral and gradient models. In this work, an integral non-local model is adopted, in which the internal dissipative non-local variable is the damage. The model is implemented, by means of user-defined subroutines, in the commercial code LS-DYNA and some examples are used to assess its performance.

2. LOCAL CONSTITUTIVE MODEL

One of the more well established phenomenological models for ductile damage in metals is based on the work of Lemaitre (Lemaitre, 1996) in which the damage variable, D, is a measure of the discontinuities density per unity area in each surface of a representative volume element at the meso-scale. Therefore, damage is a scalar variable and the model is only valid for isotropic damage in which a homogeneous distribution of micro-cavities is assumed. In order to avoid a formulation associated to each type of defect or damage growth mechanism, the prin-

ciple of strain equivalence is postulated by which is assumed that the constitutive law of the damaged material is the same as the one assumed for the undamaged material, but in which the stress tensor is substituted by the effective stress tensor defined as:

$$\tilde{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{1-D} \tag{1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor.

Lemaitre's model is based on the theory of CDM and thermodynamics of irreversible processes which strongly couples elasto-plasticity and damage at the constitutive level. In this framework, two potentials are introduced: a state potential, the Helmholtz free energy potential, which is a function of the state variables and from which is possible to define their associated variables; and a dissipation potential associated to the evolution of the state variables characterising the dissipative process. As the two dissipative processes associated to damage and plasticity result from different phenomena at the micro-scale, it is assumed that the state potential can be split in two terms as

$$\Psi = \Psi_e\left(\boldsymbol{\varepsilon}^e, D\right) + \Psi_p\left(R\right) \tag{2}$$

In this expression, where kinematic hardening and temperature effects are neglected, Ψ_e represents the specific elastic potential of the damaged material and Ψ_p is the specific plastic potential associated to hardening. In equation (2), $\boldsymbol{\varepsilon}^e$ is the elastic strain tensor and R is the isotropic hardening variable

From these potentials, it is possible to derive the state variables

$$\boldsymbol{\sigma} = \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^e};\quad \chi = \rho\frac{\partial\psi}{\partial R};\quad -Y = -\rho\frac{\partial\psi}{\partial D} \tag{3}$$

where χ is the thermodynamic force associated with the isotropic hardening variable, $-Y$ is thermodynamic force associated with damage and ρ is the density of the material.

As a consequence of the strain equivalence principle, the specific elastic potential of the damaged material is defined by

$$\Psi_e(\boldsymbol{\varepsilon}^e, D) = \frac{1}{2\rho}(1-D)\boldsymbol{\varepsilon}^e : \mathbf{D}^e : \boldsymbol{\varepsilon}^e \tag{4}$$

where $\mathbf{D}^e$ is the fourth order elasticity tensor

The Cauchy stress has then the following expression in accordance to equation (1) and the strain equivalence principle

$$\boldsymbol{\sigma} = (1-D)\mathbf{D}^e : \boldsymbol{\varepsilon}^e \tag{5}$$

The associated variable to damage is then written as

$$-Y = -\rho\frac{\partial\Psi_e}{\partial D} = \frac{1}{2}\boldsymbol{\varepsilon}^e : \mathbf{D}^e : \boldsymbol{\varepsilon}^e \tag{6}$$

or, equivalently, as

$$-Y = \frac{q^2}{6G(1-D)^2} + \frac{p^2}{2K(1-D)^2} \tag{7}$$

where p is the hydrostatic stress, q is the undamaged equivalent von Mises stress and G and K are respectively the shear and the bulk moduli.

2.1. Potentials of dissipation for plasticity and damage

For associated plasticity, the potential of dissipation is related to the particular yield (limit) function adopted. For the case of the von Mises law, coupled with the principle of strain equivalence, the potential of plastic dissipation takes the following form:

$$F_p = \frac{q}{(1-D)} - \sigma_y(\chi(R)) = \frac{\left(\frac{3}{2}\boldsymbol{s}:\boldsymbol{s}\right)^{\frac{1}{2}}}{(1-D)} - \sigma_y(\chi(R)) \tag{8}$$

where σ_y is the yield stress function and $\boldsymbol{s}$ the deviatoric stress tensor.

Likewise, it is possible to define a potential of dissipation for ductile damage in the form of the limit function:

$$F_d(-Y) = \frac{r}{(s+1)(1-D)}\left(\frac{-Y}{r}\right)^{s+1} H\left(\varepsilon^p_{acc} - \varepsilon^p_d\right) \tag{9}$$

where r is a material parameter representing the energy strength of damage and s is a non-dimensional material parameter which is a function of the temperature. The Heaviside function H of the equivalent plastic strain ε^p_{acc}, whose inclusion is justified in the next section, is defined from a critical value ε^p_d which triggers the onset of damage.

2.2. State variables evolution

From the second principle of the thermodynamics of irreversible processes, the Clausis-Duhem inequality implies that the energy of the dissipative processes of plasticity and damage must be equal or greater than zero:

$$\varphi\left(\sigma,\chi,-Y\right)=\sigma:\dot{\boldsymbol{\varepsilon}}^{p}-\chi\dot{R}\ -Y\dot{D}\geq 0 \quad (10)$$

The evolution of the internal variables may be obtained from the generalisation of the principle of maximum plastic dissipation for coupled plasticity and damage. Consequently, admitting that, for a certain level of plastic deformation $\dot{\boldsymbol{\varepsilon}}^{p}$ and a certain damage value D, among all the possible state stresses that satisfy both the plastic and damage limit functions, the solution will be obtained by maximising that energy. In Lemaitre's model, it is implied that these restrictions are imposed, resorting to only one Lagrange multiplier. This is a simplification of the model that consequently does not allow the two dissipative processes, plasticity and damage, to evolve separately. Nevertheless, this may be a valid assumption for ductile damage where this is almost always the case. That is also the reason why the Heaviside function is included in the damage limit function in order to prevent damage evolution prior to a threshold value of plastic deformation. The Lagrangian associated to this principle is then stated as:

$$L_{\varphi}(\boldsymbol{\sigma},\chi,-Y,\dot{\gamma})=\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^{p}-\chi\dot{R}-Y\dot{D}-$$
$$\dot{\gamma}\left[F_{p}(\boldsymbol{\sigma},\chi)+F_{d}(-Y)\right] \quad (11)$$

where $\dot{\gamma}$ is the Lagrange multiplier.

The evolution laws for the state internal variables may then be obtained as:

$$\frac{\partial L_{\varphi}}{\partial\sigma}=0\Leftrightarrow\dot{\varepsilon}^{p}=\dot{\gamma}\frac{\partial F_{p}}{\partial\sigma}\Leftrightarrow\dot{\varepsilon}^{p}=\frac{3}{2}\frac{\dot{\gamma}}{(1-D)}\frac{\boldsymbol{s}}{q} \quad (12)$$

$$\frac{\partial L_{\varphi}}{\partial\chi}=0\Leftrightarrow\dot{R}=-\dot{\gamma}\frac{\partial F_{p}}{\partial\chi}\Leftrightarrow\dot{R}=\dot{\gamma} \quad (13)$$

$$\frac{\partial L_{\varphi}}{\partial(-Y)}=0\Leftrightarrow\dot{D}=\dot{\gamma}\frac{\partial F_{d}}{\partial(-Y)}\Leftrightarrow$$
$$\dot{D}=\frac{\dot{\gamma}}{1-D}\left(\frac{-Y}{r}\right)^{s}H(\varepsilon_{acc}^{p}-\varepsilon_{d}^{p}) \quad (14)$$

Recurring to the previous equations and, for the case of the von Mises law, it is possible to express the equivalent plastic strain in the following form:

$$\dot{\varepsilon}_{acc}^{p}=\left(\tfrac{2}{3}\dot{\boldsymbol{\varepsilon}}^{p}:\dot{\boldsymbol{\varepsilon}}^{p}\right)^{\frac{1}{2}}=\frac{\dot{\gamma}}{1-D}\frac{\left(\frac{2}{3}\dot{\boldsymbol{\varepsilon}}^{p}:\dot{\boldsymbol{\varepsilon}}^{p}\right)^{\frac{1}{2}}}{q}=\frac{\dot{\gamma}}{1-D} \quad (15)$$

which permits to restate the previous equations that define the evolution of internal variables, for the case of coupled plasticity and damage, as:

$$\dot{\boldsymbol{\varepsilon}}^{p}=\frac{3}{2}\dot{\varepsilon}_{acc}^{p}\frac{\boldsymbol{s}}{q} \quad (16)$$

$$\dot{R}=(1-D)\dot{\varepsilon}_{acc}^{p} \quad (17)$$

$$\dot{D}=\dot{\varepsilon}_{acc}^{p}\left(\frac{-Y}{r}\right)^{s}H\left(\varepsilon_{acc}^{p}-\varepsilon_{d}^{p}\right) \quad (18)$$

3. NON-LOCAL MODELLING OF DUCTILE DAMAGE

At this point, it is important to remark that the constitutive framework presented in the last section falls within the so-called local approach. It is widely known that the standard local theory suffers from a pathological dependency on spatial discretisation when under softening regimes. Within a typical finite element framework, this corresponds to having spurious dependency on the mesh refinement used to simulate the fracturing process. This leads to unrealistic numerical solutions since the fracturing zone tends to become infinitely small as the mesh is refined.

Aiming to overcome this issue, the non-local approach has been addressed by several authors in the literature (Pijaudier-Cabot & Bažant, 1987; De Borst & Mühlhaus, 1992; De Vree et al., 1995; Strömberg & Ristinmaa, 1996; Polizzotto et al., 1998; Borino et al., 1999; Jirásek & Rolshoven, 2003; Cesar de Sa et al., 2006; Cesar de Sa & Zheng, 2007; Jirásek, 2007; Andrade et al., 2009a; among others). In the non-local theory, the constitutive behaviour is no longer independent of neighbour material points. In fact, an intrinsic length is incorporated in the continuum theory, either by means of an integral or a gradient approach. As a consequence, a diffusive effect is introduced into the model, spreading the fracturing zone over a finite area.

Over the past two and half decades, several authors have addressed non-locality as an effective tool to prevent spurious localisation and to eliminate pathological mesh dependency. After the pioneering work of Pijaudier-Cabot and Bažant (1987), who adopted a non-local formulation of integral-type for

the mesh-insensitive description of quasi-brittle materials, several relevant contributions on the subject have emerged. Initially, most of them have concentrated on the modelling of materials like concrete and soil (Pijaudier-Cabot & Bažant, 1987; De Vree et al., 1995; Peerlings et al., 1996; Benvenuti & Borino, 2002; Borino et al., 2003; among others). However, ductile materials have also, more recently, been focus of attention in many papers (Geers et al., 2003; Cesar de Sa et al., 2006; Mediavilla et al., 2006; Ricci & Brünig, 2007; Andrade et al., 2009b; Andrade et al., 2010).

In this section, we briefly revise the general concepts of non-local modelling by adopting the integral approach. Some of the recent advances accomplished within the framework of non-local models of integral-type are also highlighted and discussed. Finally, a general non-local model for elasto-plastic ductile damage is presented.

3.1. General aspects of non-locality

In a general sense, a non-local variable can be obtained by averaging the associated local quantity through the following integral

$$\bar{g}(x) = \int_V \beta(x,\xi) g(\xi) dV(\xi) \qquad (19)$$

where $\beta(x,\xi)$ is an averaging operator given by

$$\beta(x,\xi) = \frac{\alpha(x,\xi)}{\int_V \alpha(x,\zeta) dV(\zeta)} \qquad (20)$$

where $\alpha(x,\xi)$ is a weight function, here, considered to be

$$\alpha(x,\xi) = \left\langle 1 - \frac{\|x-\xi\|^2}{\ell_r^2} \right\rangle^2 \qquad (21)$$

The integral of Equation (19) incorporates a diffusive effect in the constitutive model, which prevents the chosen non-local variable to spuriously localise into a narrow band as the spatial discretisation becomes finer. Therefore, as demonstrated by Andrade et al. (2009b) and Cesar de Sa et al. (2010), only the variables that influence the softening regime should be regularised.

One important aspect of the non-local theory is the assumption of an averaging operator which is independent of the history of deformation. As discussed by Andrade et al. (2010), such hypothesis is particularly important from a computational point of view since the numerical implementation of the non-local model is much simpler if compared to the case where $\beta(x,\xi)$ is a function of deformation. As a direct consequence, the rate of the averaging operator vanishes, i.e.,

$$\frac{\partial \beta(x,\xi)}{\partial t} = \dot{\beta}(x,\xi) = 0 \qquad (22)$$

Thus, the rate of the chosen non-local variable is simply given by

$$\dot{\bar{g}}(x) = \int_V \beta(x,\xi) \dot{g}(\xi) dV(\xi) \qquad (23)$$

At this point, some interpretations and conclusions can be withdrawn from the above statements. For instance, if the history of deformation has no influence on the non-local averaging operator, it means that the intrinsic length remains constant as the body undergoes deformation. Thus, the size of the fracturing area is inherently assumed to be constant and dictated by ℓ_r. However, there is no experimental evidence supporting this statement. Apparently, a more realistic modelling would consider an evolving intrinsic length, for which ℓ_r would be a function of other variables that are known to significantly influence the dissipative fracturing process, e.g., the plastic strain, the stress state, the internal degradation or the history of deformation itself. On the other hand, the consideration of a non-constant intrinsic length as well as an averaging operator which is dependent on the history of deformation is still very challenging, demanding further developments both on theoretical and computational point of view of the non-local theory. Nonetheless, the present non-local theory has already been successfully employed in the task of reducing pathological mesh dependency and provides more sound results than the standard local theory.

It is also important to mention that the assumption of a deformation-independent averaging operator facilitates the extension of the non-local theory to finite strains. This point has been discussed in more detail by Andrade et al. (2010), where different averaging strategies at finite strains have been presented. Therefore, in the following, we will limit ourselves to show the theory in small strains in order to keep notation simple. However, the extension to the finite strain domain can be straightforwardly done by employing the same procedures adopted by Andrade et al. (2010).

3.2. Non-local elasto-plastic damage model

In the works of Andrade et al. (2009b) and Cesar de Sa et al. (2010), different non-local variables have been chosen to enhance the local continuum. The results have shown that damage is a good candidate, especially due to the fact that it highly influences the softening regime. Therefore, we will consider herein the choice of damage as the non-local variable for the definition of the non-local model. The elastic-damage constitutive relation can then be expressed as

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \left(1-\overline{D}(\boldsymbol{x})\right)\mathbf{D}^{e}(\boldsymbol{x}) : \boldsymbol{\varepsilon}^{e}(\boldsymbol{x}) \qquad (24)$$

The yield function, the plastic flow rule and the evolution of isotropic hardening are also rewritten, respectively given by

$$F_p(\boldsymbol{x}) = \frac{q(\boldsymbol{x})}{1-\overline{D}(\boldsymbol{x})} - \sigma_y(R(\boldsymbol{x})) \qquad (25)$$

$$\dot{\boldsymbol{\varepsilon}}^{p}(\boldsymbol{x}) = \frac{3}{2}\frac{\dot{\gamma}(\boldsymbol{x})}{\left(1-\overline{D}(\boldsymbol{x})\right)}\frac{\boldsymbol{s}(\boldsymbol{x})}{q(\boldsymbol{x})} \qquad (26)$$

and

$$\dot{R}(\boldsymbol{x}) = \dot{\gamma}(\boldsymbol{x}) \qquad (27)$$

In contrast to the model of Section 2, all constitutive variables have been explicitly written a function of their spatial location, generically represented by $\boldsymbol{x}$. This stems from the fact that the non-local formulation yields on an enriched continuum theory for which the position of the material point in the body is essential for the determination of its constitutive behaviour. Moreover, since non-local damage, $\overline{D}(\boldsymbol{x})$, is now the actual state variable, its rate should play the role of damage evolution. From equations (22) and (23), we know that

$$\dot{\overline{D}}(\boldsymbol{x}) = \int_V \beta(\boldsymbol{x},\boldsymbol{\xi})\dot{D}(\boldsymbol{\xi})dV(\boldsymbol{\xi}) \qquad (28)$$

where

$$\dot{D}(\boldsymbol{\xi}) = \frac{\dot{\gamma}(\boldsymbol{\xi})}{1-\overline{D}(\boldsymbol{\xi})}\left(\frac{-Y(\boldsymbol{\xi})}{r}\right)^{s} \qquad (29)$$

where $\boldsymbol{\xi}$ corresponds to the global coordinate of a given surrounding point in the vicinity of $\boldsymbol{x}$. It is worth mentioning that the Heaviside function has not been considered in equation (29). However, its inclusion in the constitutive model is rather straightforward.

Finally, the Kuhn-Tucker conditions ($\dot{\gamma}(\boldsymbol{x}) \geq 0$, $F_p(\boldsymbol{x}) \leq 0$ and $\dot{\gamma}(\boldsymbol{x})F_p(\boldsymbol{x}) = 0$) must be fulfilled, leading to the final constitutive model, summarised in table 1. It is important to remark that, since every material point depends on its neighbourhood, the Kuhn-Tucker conditions must, in practice, simultaneously hold for all points of the whole body. As a matter of fact, the integral character of the non-local model yields on a substantially more complicated constitutive problem for which analytical solutions are difficult to obtain. A strategy that numerically solves the non-local material problem through a global integration algorithm has been addressed by Andrade et al. (2010). In Section 4, we will present an alternative implementation strategy suitable for explicit formulations.

4. NUMERICAL IMPLEMENTATION

4.1. Local damage model

One of the possible modifications for Lemaitre's damage model is achieved by disregarding the effects of kinematic hardening. As shown by de Souza Neto (2002), the consideration only of isotropic hardening leads to a remarkably simple and efficient numerical algorithm.

The algorithm for the numerical integration of the constitutive model starts with the definition of the typical *elastic trial state*. Notice that the constitutive behaviour is meant to be locally computed at *every material point* within a generic time interval $[t_n, t_{n+1}]$, where the constitutive variables $\boldsymbol{\varepsilon}_n^p$, $\boldsymbol{\sigma}_n$, R_n and D_n are known 'a priori'. The goal of the algorithm is to find the updated values of $\boldsymbol{\varepsilon}_{n+1}^p$, $\boldsymbol{\sigma}_{n+1}$, R_{n+1} and D_{n+1} for a given strain increment $\Delta\boldsymbol{\varepsilon}$ within $[t_n, t_{n+1}]$. In a typical finite element framework, the material points correspond to the Gauss integration points.

The elastic trial strain tensor is given by

$$\boldsymbol{\varepsilon}_{n+1}^{e\ trial} = \boldsymbol{\varepsilon}_n^{e} + \Delta\boldsymbol{\varepsilon} \qquad (30)$$

from which it is possible to compute the elastic trial stress tensor, written as.

$$\widetilde{\boldsymbol{\sigma}}_{n+1}^{e\ trial} = \mathbf{D}^{e} : \boldsymbol{\varepsilon}_{n+1}^{e\ trial} \qquad (31)$$

Table 1. *Classical non-local damage model with* $\overline{D}$

(i) Strain tensor additive split	$\boldsymbol{\varepsilon}(\boldsymbol{x}) = \boldsymbol{\varepsilon}^e(\boldsymbol{x}) + \boldsymbol{\varepsilon}^p(\boldsymbol{x})$
(ii) Elastic law	$\boldsymbol{\sigma}(\boldsymbol{x}) = \left(1-\overline{D}(\boldsymbol{x})\right)\mathbf{D}^e(\boldsymbol{x}) : \boldsymbol{\varepsilon}^e(\boldsymbol{x})$
(iii) Yield criterion	$F_p(\boldsymbol{x}) = \dfrac{q(\boldsymbol{x})}{1-\overline{D}(\boldsymbol{x})} - \sigma_y(R(\boldsymbol{x}))$
(iv) Plastic flow	$\dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{x}) = \dfrac{3}{2}\dfrac{\dot{\gamma}(\boldsymbol{x})}{\left(1-\overline{D}(\boldsymbol{x})\right)}\dfrac{\boldsymbol{s}(\boldsymbol{x})}{q(\boldsymbol{x})}$
(v) Evolution of isotropic hardening	$\dot{R}(\boldsymbol{x}) = \dot{\gamma}(\boldsymbol{x})$
(vi) Evolution of damage	$\dot{\overline{D}}(\boldsymbol{x}) = \int_V \beta(\boldsymbol{x},\boldsymbol{\xi})\dot{D}(\boldsymbol{\xi})dV(\boldsymbol{\xi})$ where $\dot{D}(\boldsymbol{\xi}) = \dfrac{\dot{\gamma}(\boldsymbol{\xi})}{1-\overline{D}(\boldsymbol{\xi})}\left(\dfrac{-Y(\boldsymbol{\xi})}{r}\right)^s$
(vii) Kuhn-Tucker conditions	$\dot{\gamma}(\boldsymbol{x}) \geq 0\,;\; F_p(\boldsymbol{x}) \leq 0\,;$ $\dot{\gamma}(\boldsymbol{x})F_p(\boldsymbol{x}) = 0$

Considering now the deviatoric/hydrostatic split of the stress tensor, the elastic trial stress tensor is alternatively written as

$$\widetilde{\boldsymbol{\sigma}}^{\,\mathrm{e\,trial}}_{\mathrm{n+1}} = \widetilde{\boldsymbol{s}}^{\,\mathrm{trial}}_{\mathrm{n+1}} + \widetilde{p}^{\,\mathrm{trial}}_{\mathrm{n+1}}\boldsymbol{I} \tag{32}$$

where $\widetilde{\boldsymbol{s}}^{\,trial}_{n+1}$ and $\widetilde{p}^{\,trial}_{n+1}$ are the *effective* trial deviatoric and hydrostatic stresses, respectively given by

$$\widetilde{\boldsymbol{s}}^{\,\mathrm{trial}}_{\mathrm{n+1}} = 2G\boldsymbol{\varepsilon}^{\,\mathrm{e\,trial}}_{\mathrm{d\,n+1}}, \quad p^{\,\mathrm{trial}}_{\mathrm{n+1}} = K\varepsilon^{\,\mathrm{e\,trial}}_{\mathrm{v\,n+1}}, \tag{33}$$

where

$$\boldsymbol{\varepsilon}^{e\ trial}_{d\ n+1} = \boldsymbol{\varepsilon}^{e}_{d\ n} + \Delta\boldsymbol{\varepsilon}_d; \qquad \varepsilon^{e\ trial}_{v\ n+1} = \varepsilon^{e}_{v\ n} + \Delta\varepsilon_v, \tag{34}$$

in which the strain deviator and the volumetric strain have been denoted, respectively, by $\boldsymbol{\varepsilon}_d$ and ε_v.

Finally, with equation (33) it is possible to compute the elastic trial von Mises equivalent stress from the *effective* deviatoric stress as

$$\widetilde{q}^{\,trial}_{n+1} = \sqrt{\frac{3}{2}}\left\|\widetilde{\boldsymbol{s}}^{\,trial}_{n+1}\right\| \tag{35}$$

where $\tilde{q}^{\,trial}_{n+1}$ is necessary for the proper evaluation of the yield function and to check whether the pseudo-increment is elastic or plastic.

Following standard procedures of elastic predictor/return mapping schemes, Lemaitre's constitutive model can be written in its (pseudo-)time-discretised version by the following system of equations

$$\begin{cases} \boldsymbol{\varepsilon}^{e}_{n+1} = \boldsymbol{\varepsilon}^{e\ trial}_{n+1} - \dfrac{3}{2}\dfrac{\Delta\gamma}{\left(1-D_{n+1}\right)}\dfrac{\boldsymbol{s}_{n+1}}{q_{n+1}} \\ R_{n+1} = R_n + \Delta\gamma \\ D_{n+1} = D_n + \dfrac{\Delta\gamma}{1-D_{n+1}}\left(\dfrac{-Y_{n+1}}{r}\right)^s \\ \dfrac{q_{n+1}}{1-D_{n+1}} - \sigma_y\left(R_{n+1}\right) = 0 \end{cases} \tag{36}$$

where $\boldsymbol{\varepsilon}^{e}_{n+1}$, R_{n+1}, $\Delta\gamma$ and D_{n+1} are the unknowns of the incremental initial boundary value constitutive problem. It is important to remark that the last equation of the system is the *consistency condition* which, in practice, acts as a constraint for the constitutive problem.

The above system of equations is unattractive from the numerical point of view due to the high computational burden if compared to simpler elasto-plastic models (e.g. von Mises isotropic plasticity). However, by performing some relatively straightforward operations, it is possible to reduce the system of equation (36) to a *single* scalar non-linear equation, which can be written as

$$F(\Delta\gamma) = 3G\Delta\gamma - (1-D_n)(\widetilde{q}^{\,trial}_{n+1} - \sigma_y) + \frac{(\widetilde{q}^{\,trial}_{n+1} - \sigma_y)^2}{3G}\left(\frac{-Y(\Delta\gamma)}{r}\right)^s \tag{37}$$

where $\Delta\gamma$ is now the only unknown. In fact, the scalar equation above is much simpler to be worked out, significantly reducing the computational cost per Gauss point. The derivation of equation (37) is omitted here for convenience and the reader is referred to de Souza Neto (2002) for further details. The complete stress update algorithm for the efficient numerical integration of Lemaitre's simplified model by means of a fully implicit elastic predictor/return mapping scheme (Simo & Hughes, 1998) is summarised in table 2 in pseudo-code format.

4.2. Non-local damage models

The numerical implementation of non-local models of integral-type has been usually considered as one of the main disadvantages of this kind of regularised model. The main reason for that is the presence of the integral in the constitutive evolution equations that prevents the model to have the consis-

tency condition fulfilled locally. Some implicit gradient-dependent elasto-plastic models proposed in the literature have tackled the problem by adding the non-local variable in the global structural problem and then satisfying the consistency condition only in a weak sense (Peerlings et al., 1996; Geers et al., 2003). Adopting the integral-type theory, other authors have employed a solution scheme where all integration points are solved simultaneously (Strömberg & Ristinmaa, 1996; Andrade et al., 2009a, 2009c, 2010; Cesar de Sa, 2010).

In the particular case of elasto-plastic damage models, the works of Cesar de Sa et al. (2010) and Andrade et al. (2009a, 2009c, 2010) have presented detailed algorithms for the numerical implementation of non-local models of integral-type. The strategy was based on a global version of the typical return-mapping scheme, for which all the integration points are integrated simultaneously. This assures the consistency condition to be fulfilled at all material points at the end of a given deformation step. Within an elasto-plastic damage framework, this algorithm has proved very efficient and relatively simple to implement in existing finite element codes.

***Table 2**. Stress update algorithm for Lemaitre's simplified (local) model.*

(i) Define elastic trial state

$$\boldsymbol{\varepsilon}^{e\ trial}_{n+1} = \boldsymbol{\varepsilon}^{e}_{n} + \Delta\boldsymbol{\varepsilon}; \qquad \widetilde{\boldsymbol{s}}^{trial}_{n+1} = 2G\boldsymbol{\varepsilon}^{e\,trial}_{d\,n+1}$$

$$R^{trial}_{n+1} = R_n;$$

$$\widetilde{p}^{trial}_{n+1} = K\varepsilon^{e\,trial}_{v\,n+1}; \qquad \widetilde{q}^{trial}_{n+1} = \sqrt{\frac{3}{2}}\left\|\widetilde{\boldsymbol{s}}^{trial}_{n+1}\right\|$$

(ii) Check plastic admissibility

IF $\widetilde{q}^{trial}_{n+1} - \sigma_y\left(R^{trial}_{n+1}\right) \le 0$ THEN

SET $(\cdot)_{n+1} = (\cdot)^{trial}_{n+1}$ and EXIT

ENDIF

(iii) Solve the single residual equation

$$F(\Delta\gamma) = 3G\Delta\gamma - (1-D_n)(\widetilde{q}^{trial}_{n+1} - \sigma_y) + \frac{(\widetilde{q}^{trial}_{n+1} - \sigma_y)^2}{3G}\left(\frac{-Y}{r}\right)^s$$

for $\Delta\gamma$ with the Newton-Raphson method where

$$-Y(\Delta\gamma) = \frac{\left(\sigma_y\left(R_n + \Delta\gamma\right)\right)^2}{6G} + \frac{\left(\widetilde{p}^{trial}_{n+1}\right)^2}{2K}$$

(iv) Update stress and internal variables

$$D_{n+1} = 1 - \left(\frac{3G\Delta\gamma}{\widetilde{q}^{trial}_{n+1} - \sigma_y\left(R_{n+1}\right)}\right) \qquad \boldsymbol{s}_{n+1} = \frac{q_{n+1}}{\widetilde{q}^{trial}_{n+1}}\widetilde{\boldsymbol{s}}^{\boldsymbol{trial}}_{\boldsymbol{n+1}};$$

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{s}_{n+1} + p_{n+1}\boldsymbol{I};$$

$$p_{n+1} = (1-D_{n+1})\widetilde{p}^{trial}_{n+1}; \qquad \boldsymbol{\varepsilon}^{e}_{n+1} = \frac{1}{2G}\boldsymbol{s}_{n+1} + \frac{1}{3}\varepsilon^{e\ trial}_{v\ n+1}\boldsymbol{I}$$

$$q_{n+1} = (1-D_{n+1})\sigma_y\left(R_{n+1}\right);$$

(v) EXIT

However, for its implementation, one needs to have access to all integration points of the finite element mesh in order to proceed with the global integration scheme. In general, this fact does not represent a problem if one has full access to the source code of the finite element program, or, at least, access to the algorithm that assembles the internal force vector. Unfortunately, most commercial FE codes do not allow the user to do the necessary modifications since they restrict the user-defined constitutive model to be computed for a generic Gauss point only.

The global integration approach can be avoided by substituting it by alternative formulations that only approximate the non-local theory. For example, Tvergaard and Needleman (1995) have proposed a non-local model intended for the description of ductile materials where they have approximated the rate of the non-local variable (in their case, the porosity f) by

$$\dot{\bar{f}}^{\,t_{n+1}} = K^{nl}\,\dot{f}^{\,t_{n+1}} \tag{38}$$

where $\dot{f}$ and $\dot{\bar{f}}$ are, respectively, the local and non-local rate of the porosity and K^{nl} is a penalty factor given by

$$K^{nl} = \frac{\dot{\bar{f}}^{\,t_n}}{\dot{f}^{\,t_n}} \tag{39}$$

where $\dot{f}^{\,t_n}$ and $\dot{\bar{f}}^{\,t_n}$ are the local and non-local rate of the porosity at the last converged incremental step. As a matter of fact, this methodology can be interpreted as an "explicit non-local integration" since only information of the previous step is used for the computation of the non-local quantity. The main advantage of this strategy is that the state update procedure keeps its typical local format. On the other hand, a major restriction arises since the time-step must be kept small enough to avoid instabilities in the solution. In explicit codes, however, the time-step size has necessarily to be small in order to render stable solutions where a critical time-step is dictated by the size of the smallest element of the mesh. Thus, in this case, the use of an approximate expression like the one in equation (38) is very attractive from a computational point of view since the stress-update can be done locally and the time-step will be necessarily kept small either way. Following these ideas, we have implemented a similar non-local strategy in the commercial FE program LS-DYNA.

The implementation is described in detail in the following.

To start with, an expression equivalent to the one in equation (38) can be written for the non-local model described in Section 3. Therefore, non-local damage is now given by

$$\dot{\bar{D}}^{t_{n+1}} = K^{nl}\dot{D}^{t_{n+1}} = \frac{\dot{\bar{D}}^{t_n}}{\dot{D}^{t_n}}\dot{D}^{t_{n+1}} \tag{40}$$

It is important to remark that the relation above must be consistently introduced in the local stress update algorithm. When the evolution of the chosen non-local variable is explicitly given in the numerical implementation of the local constitutive model, the modification is rather straightforward. However, when algebraic manipulations have been previously done, one must take special care in order to correctly add the penalty factor in the residual equations. For the case of the local damage algorithm presented in Section 4.1, the single residual equation must be modified to

$$F(\Delta\gamma) = 3G\Delta\gamma - (1-D_n)(\tilde{q}_{n+1}^{trial} - \sigma_y) + K^{nl}\frac{(\tilde{q}_{n+1}^{trial} - \sigma_y)^2}{3G}\left(\frac{-Y}{r}\right)^s \tag{41}$$

In addition, the rate of damage must be stored after convergence has been achieved with the local material integration algorithm. No further modifications are necessary in the stress update algorithm.

However, one still needs to compute the non-local counterpart of the damage rate, which requires having the values of the local damage rate of the surrounding points. In fact, this step may still be very complex depending on the commercial FE program used. In the case of LS-DYNA, the access to information of other elements is available, especially because the software allows the user to create vectorial implementations. Therefore, the arrays containing the stress, plastic strain, and the history variables of the elements of the mesh are readily available in the user subroutine `urmathn` (in the case of solids) and have the dimension of `nlq`, which is a parameter dependent on the machine architecture. Whenever the software enters the subroutine `urmathn`, the information of a given group of elements (the size of the group depends on the parameter `nlq`) is supplied. The user subroutine is then called as many times as necessary until the material integration algorithm of all elements of that particular group has been carried out. For instance, if `nlq` is defined to be 96 and the mesh has 2000 solid elements, the subroutine `urmathn` will be accessed 21 times by LS-DYNA, where at each time the arrays `sigX`, `epsps` and `hsvs` will store, respectively, the stresses, the plastic strains and the history variables of a different group of 96 elements. This code structure can be conveniently used for the implementation of the non-local strategy since we need the rates of the surrounding points only at the previous converged time. Therefore, a general procedure can be adopted as follows.

In the first incremental step, all the information necessary (nodal coordinates, connectivities, etc.) for the computation of the non-local factors, β_{ij}, can be retrieved and stored in user-defined arrays. Meanwhile, the material integration is done locally for all elements of the mesh and, after convergence of the local material algorithm, the rate of chosen non-local variable (in the present case, damage) is stored in a user-defined array. This information will be necessary in the subsequent steps when the non-local averages for the computation of the penalty factor, K^{nl}, will be updated. In turn, the penalty factor is input as an argument of the local material user routine. In order to avoid numerical instabilities, the penalty factor is set to 1.0 if the local rate of damage is less than a given prescribed value (e.g. 10^{-12}).

A critical step in the procedure described above is the correct storage of the user-defined arrays. To prevent data loss and to assure that the information computed in the previous step will be available in the next one, the FORTRAN77 `save` statement should be issued at the beginning of the subroutine `urmathn` for the user-defined arrays.

In order to know the current step (or cycle, as it is called in LS-DYNA), the following common block needs to be included at the beginning of the subroutine `urmathn`:

```
common/bk06/idmmy,iaddp,ifil,
  maxsiz,ncycle,time(2,30)
```

The variable `ncycle` is an integer that stores the number of the current step. For the determination of the non-local averaging factors, as well as the element connectivities and the nodal coordinates, it is in general easier to handle with the external ID's, which can be easily retrieved through the function `lqfinv(id_internal,itype)`.

Table 3. *FORTRAN code excerpt for the implementation of the non-local model in LS-DYNA for solid elements*

```
subroutine urmathn (...)
....
common/bk06/idmmy,iaddp,ifil,maxsiz, ncycle,
       time(2,30)
....
dimension betaij(...), dot_nlv(mxelem,2)
dimension connect(mxelem,8), coord(mxnode,3)
....
C Save data in arrays for the next step
save betaij, dot_nlv, connect, coord
....
if(ncycle.eq.1.and.nnm1.eq.0)then
C Initialise rates for penalty factor calculation
  do ielem=1,nelem
  dot_nlv(ielem,1)=0.0    ! local rate
  dot_nlv(ielem,2)=0.0    ! non-local rate
  enddo
endif
if(ncycle.eq.1)then
C Get connectivities (e.g. linear hexahedron)

  do i=lft,llt
  ielem=lqfinv(nnm1+i,2)
  connect(ielem,1)=lqfinv(ix1(i),1)
  connect(ielem,2)=lqfinv(ix2(i),1)
  connect(ielem,3)=lqfinv(ix3(i),1)
  connect(ielem,4)=lqfinv(ix4(i),1)
  connect(ielem,5)=lqfinv(ix5(i),1)
  connect(ielem,6)=lqfinv(ix6(i),1)
  connect(ielem,7)=lqfinv(ix7(i),1)
  connect(ielem,8)=lqfinv(ix8(i),1)
  enddo
C Get coordinates
  call get_coordinates (..., coord , r_mem(dm_x)
       ,...)
endif
if(ncycle.eq.2.and.nnm1.eq.0)then
  call compute_non-local_factors (..., betaij ,
       connect , coord ,...)
endif
if(ncycle.ge.2.and.nnm1.eq.0)then
  call compute_non-local_rate (..., betaij ,
       dot_nlv ,...)
endif
....
do 90 i=lft,llt
....
C Get global element ID
  ielem=lqfinv(nnm1+i,2)
C Compute penalty factor
  penalty_factor=dot_nlv(ielem,2)/
dot_nlv(ielem,1)
....
C Call local user material routine
41 call umat41 (..., penalty_factor , rate_local
       ,...)
....
C Store local rate of damage
  dot_nlv(ielem,1)=rate_local
....
90 continue
....
end subroutine urmathn
```

A FORTRAN code excerpt schematically depicting the implementation procedure is given in table 3 for reference. Note that only the main steps for solid elements have been presented. Nevertheless, the same concepts can be straightforwardly extended for shell elements (in subroutine `urmats`).

5. NUMERICAL EXAMPLES

5.1. Axisymmetric specimen

In this first example, the analysis of an axisymmetric specimen (see figure 1) is carried out in order to assess the regularising effects of the proposed numerical implementation. A similar specimen has been investigated by the authors in previous contributions (Andrade et al., 2009c, 2010) using an implicit in-house finite element code. Three mesh refinements using linear quadrilateral elements with reduced integration (see figure 1) have been considered in order to capture the pathological mesh dependency. The material properties adopted are given in table 4.

Table 4. *Material properties for the axisymmetric example*

Property	Value
Young's modulus	$E = 210000$ MPa
Poisson's ratio	$\nu = 0.3$
Damage exponent	$s = 1.0$
Damage denominator	$r = 2.5$ MPa
Initial yield stress	$\sigma_{yo} = 80.559$ MPa
Yield hardening curve	$\sigma_y = \sigma_{yo} + \left[589\left(10^{-4} + R\right)^{0.216} - 80.559\right]$ MPa
Non-local characteristic length	$\ell_r = 1.0$ mm

It is worth mentioning that the regularising effects of the non-local theory can only be achieved if the characteristic length is large enough to span at least some elements. Therefore, in the present case, the value of $\ell_r = 1.0$ mm has been chosen aiming to fulfil this condition rather than based on experimental measurements. Nonetheless, this consideration is reasonable enough to assess the non-local algorithm.

The specimen is loaded at its ending edge with a low velocity that grows linearly with time, therefore, minimising the effects of inertia. Figure 2 shows the damage contours for the local case. We notice that damage tends to concentrate at the critical element meanwhile its numerical value also tends to increase. Conversely, the non-local solution provides a different behaviour. Observing figure 3, it is possible to notice that both the damaging area and the damage values are kept nearly constant upon mesh refinement.

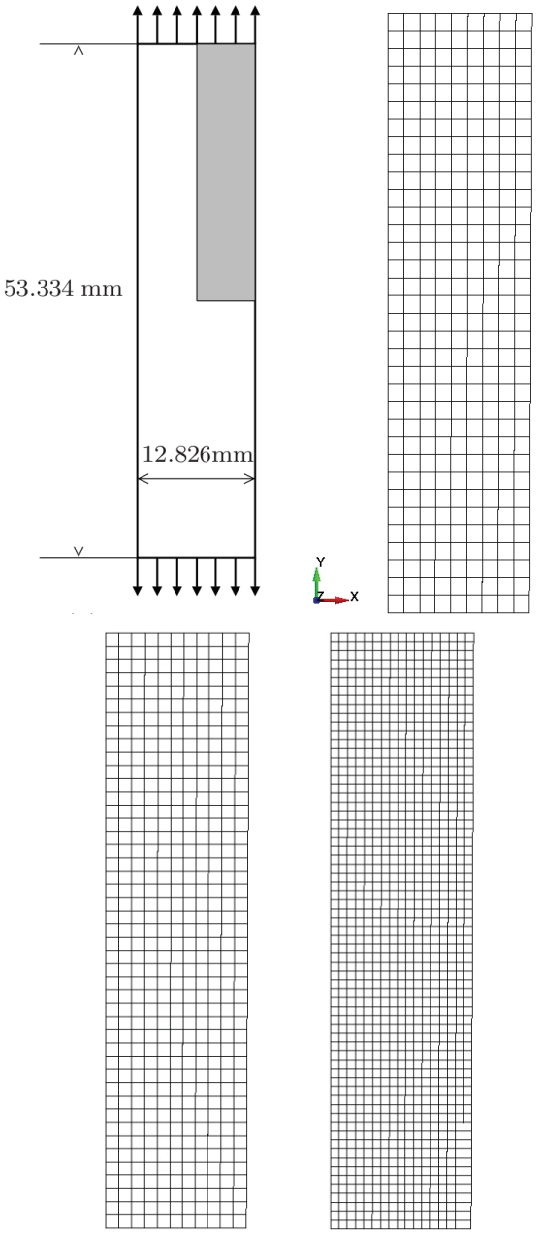

Fig. 1. *Geometry and different mesh refinements for the axi-symmetric specimen.*

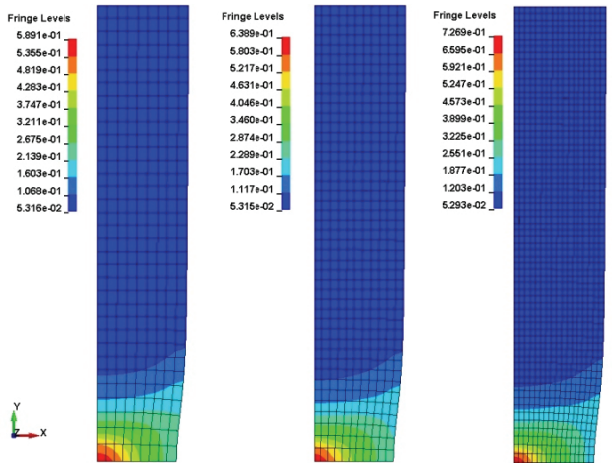

Fig. 2. *Damage contours for the local case.*

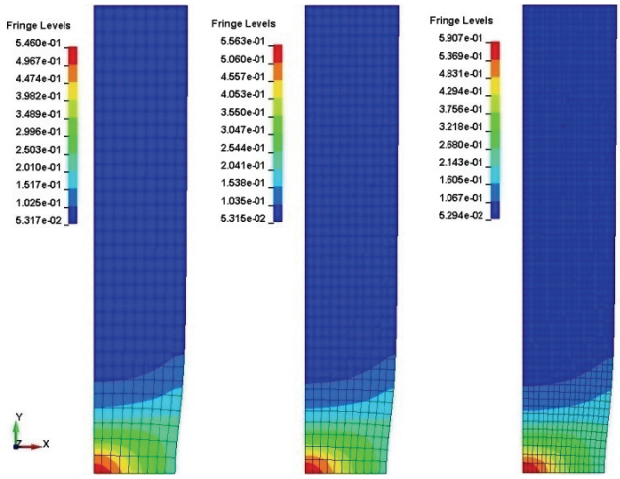

Fig. 3. *Damage contours for the non-local case.*

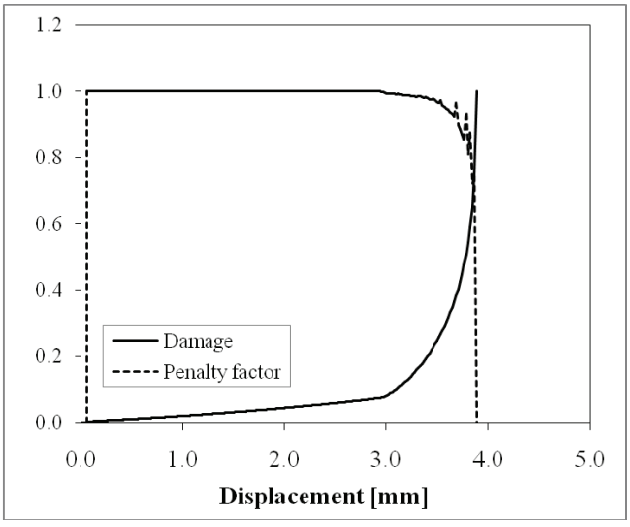

Fig. 4. *Evolution of damage and penalty factor at the critical point of the finest mesh (non-local solution).*

In figure 4, the evolution of the damage variable and of the non-local penalty factor, K^{nl}, are plotted at the critical point of the finest mesh. At the initial stages, when damage is still low, the penalty factor is kept constant and equal to the unity. However,

when damage begins to evolve rapidly, the penalty factor decreases accordingly, which slows down the rate of damage evolution. As a consequence, spurious localisation is avoided.

5.2. Sheet forming

This example is based on the analysis carried out by de Souza Neto et al. (1994). It consists of the simulation of a sheet metal forming process (see figure 5). Only one quarter of the geometry has been simulated. Despite the fact that this kind of process usually requires anisotropic constitutive laws for an accurate prediction, the use of an isotropic model is helpful and sufficient to demonstrate how the proposed non-local implementation can significantly alleviate the pathological mesh dependency issue. Nonetheless, it is worth mentioning that the same non-local methodology can be extended to account for anisotropy. The sheet is considered to be made of an aluminium alloy, while the punch and the die are regarded as perfect rigid materials. All the material properties employed are given in table 5. Different mesh refinements using linear shell elements with 5 integration points have been used to model the sheet as shown in figure 6. A general surface-to-surface contact algorithm available in LS-DYNA has been employed to consider the frictional contact between the parts. Both static and dynamic friction coefficients were assumed to be equal to 0.05. The dynamic effects have been minimised by applying a linearly growing low velocity on the punch. Similar to the last example, a characteristic length that is sufficiently large to get the influence of enough surrounding elements has been adopted.

Table 5. *Material properties for the sheet forming example*

Property	Value
Young's modulus	$E = 69004$ MPa
Poisson's ratio	$\nu = 0.3$
Damage exponent	$s = 1.0$
Damage denominator	$r = 2.5$ MPa
Initial yield stress	$\sigma_{yo} = 80.559$ MPa
Yield hardening curve	$\sigma_y = \sigma_{yo} + \left[589\left(10^{-4} + R\right)^{0.216} - 80.559\right]$ MPa
Non-local characteristic length	$\ell_r = 5.0$ mm

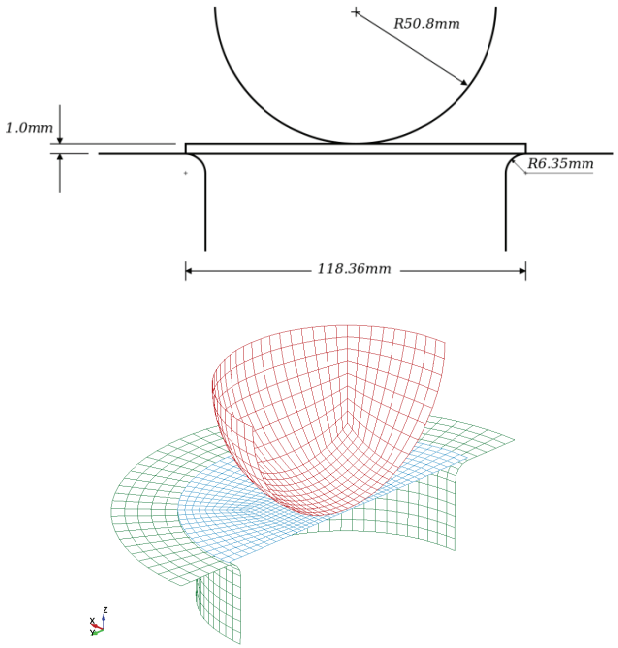

Fig. 5. *Simulation of the sheet forming process: geometry and finite element model for the coarsest mesh (the model has been mirrored in respect to the XZ plane).*

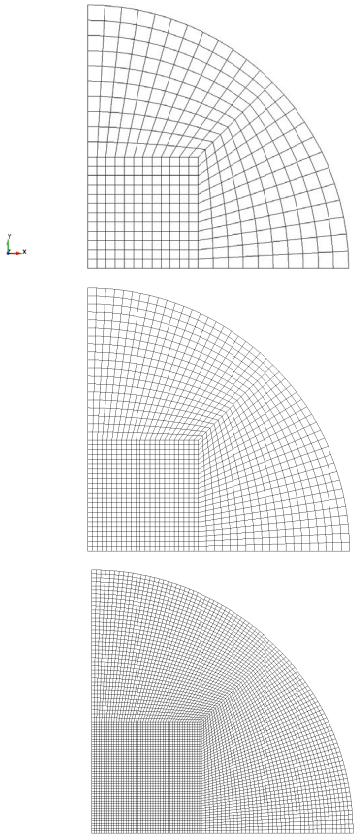

Fig. 6. *Mesh refinements of the sheet for the metal forming simulation (top view).*

In figure 7, the damage contours for the local solution are plotted for the same punch displacement. Clearly, damage tends to localise very rapidly upon mesh refinement. On the other hand, when the non-

local solution is adopted, the localisation effects are significantly attenuated since the damage values do not vary as much as in the local case (see figure 8). Furthermore, the differences on the contours are also smaller, especially on the zone where damage concentrates.

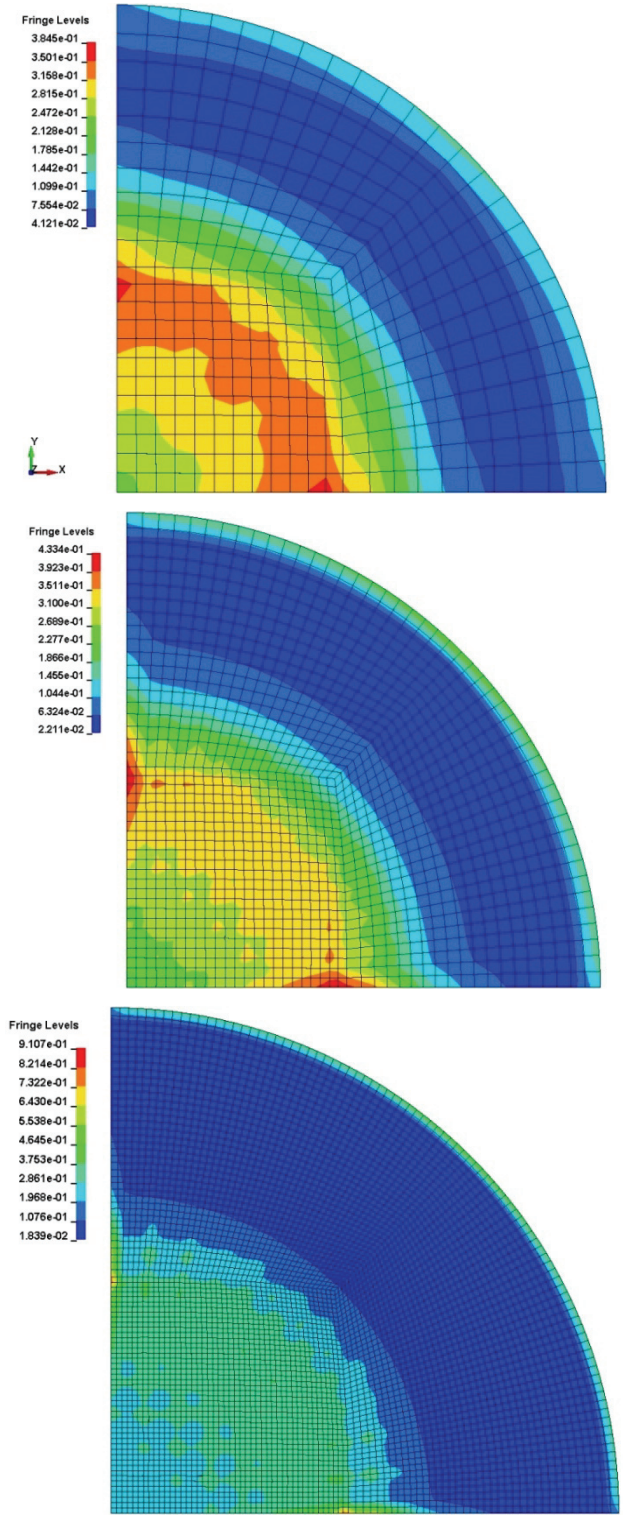

Fig. 7. *Damage contours for the local case (top view).*

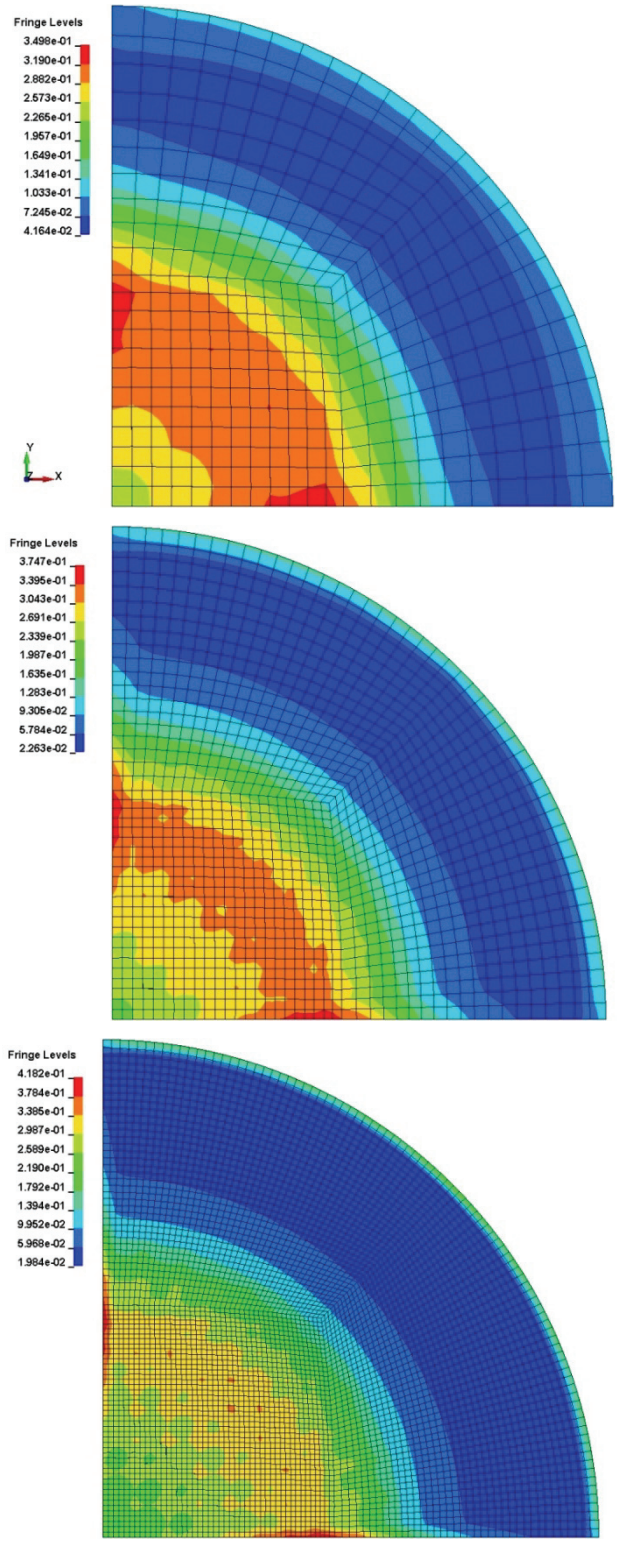

Fig. 8. *Damage contours for the non-local case (top view).*

6. CONCLUSIONS

A general overview of the general framework for the description of ductile damage has been addressed in this paper. The standard local theory has been revised where some important interpretations have been highlighted. The general aspects of the non-local theory incorporated into a continuum damage

framework have also been focus of discussion where some of the main assumptions have been reviewed and clarified. The numerical implementation of an alternative non-local formulation has been presented in detail for the finite element code LS-DYNA. The results of two sample cases have shown that the methodology was able to alleviate the pathological mesh dependency observed due to the softening regime caused by the damage model. However, since the proposed non-local formulation is only an approximation of the true non-local theory, it is still not clear if the methodology will properly prevent localisation for a wider number of cases. Therefore, a deeper assessment of the presented non-local strategy, together with element erosion techniques under several circumstances, should be focus of further investigation.

ACKNOWLEDGEMENTS

F.X.C. Andrade was supported by the Programme AlBan, the European Union Programme of High Level Scholarships for Latin America, scholarship no. E07D401751BR.

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PROBLEMY TEORETYCZNEJ I NUMERYCZNEJ ANALIZY PLASTYCZNEGO PĘKANIA – WYBRANE ZAGADNIENIA

Streszczenie

Celem pracy jest przedstawienie przeglądu najnowszych osiągnięć w zakresie opisu plastycznego pękania, zarówno od strony teoretycznej jak i numerycznej. Na wstępie omówiono klasyczną lokalną teorię odnoszącą się do termodynamiki nieodwracalnych procesów, wykorzystywaną do opisu ogólnego modelu sprężysto-plastycznego pękania. Naświetlono również założenia i ograniczenia klasycznej teorii pojawiające się wtedy, kiedy równania konstytutywne są uzyskiwane z rozwiązania problemu minimalizacji z ograniczeniami. Omówiono też ostatnie osiągnięcia w zakresie nie lokalnego modelowania sprężysto-plastycznego pękania, pokazując zasady i konsekwencje wynikające z nie lokalnego traktowania tego procesu. Zagadnienia związane z efektywnym numerycznym opisem lokalnej i nielokalnej teorii są również omówione w pracy, a specjalną uwagę poświęcono implementacji modeli nie lokalnych. Nowa strategia obliczeniowa, umożliwiająca implementację tych modeli w komercyjnych programach symulacyjnych, została szczegółowo przedstawiona na przykładzie LS-DYNA. Fragmenty kodu w języku FORTRAN, w których pokazano główne kroki implementacji, są przytoczone w pracy. Efektywność nie lokalnego modelu jest oceniona na podstawie symulacji odkształcania osiowosymetrycznych próbek oraz procesów tłoczenia. Wykazano, że w obydwóch analizowanych przypadkach nielokalna strategia numeryczna pozwala na obniżenie wrażliwości rozwiązania na rozmiar siatki, które jest nierozerwalnie związane ze sprężysto-plastycznymi modelami pękania.

Received: July 31, 2010
Received in a revised form: October 10, 2010
Accepted: November 14, 2010