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ELASTO-PLASTIC MATERIAL MODEL WITH DAMAGE PARAMETER IN METAL FATIGUE

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Abstract

In this paper the numerical method for prediction of fatigue life of structure is presented. The finite element modelling and damage parameter calculation are based on the algorithm described originally by Chaboche (1987), Luccioni et al. (1996) and Oller et al. (2005). This algorithm is simplified, i.e. the Goodman relationships between the mean and stress amplitude are included. It is extended and applied for simulation of crack propagation in the rotor shaft. The material constitutive model includes plastic effects and damage. The S–N fatigue functions taken from experiment are modified so as to be dependent on the real ratio between minimum and maximum stress in the critical zones. So the coupling between damage, plasticity and fatigue is taken into account.

Key words: metal fatigue, damage mechanics

1. INTRODUCTION

Fatigue phenomena can be defined as a process of permanent and progressive degradation of the material under cycling loading. A progressive loss of material strength is localized and depends on the number of load cycles, stress amplitude, stress ratio. The loss of material strength can be interpreted as micro cracking (taking into account local plastic effects) which can evaluate to macro cracks and lead to final collapse of the structure. In this paper fatigue phenomena is modelled by damage mechanics extending the work of Chaboche (1987) and Oller (2005).

2. THERMODYNAMICAL BASICS

The inelastic theories of damage and plasticity allow to solve mechanical problems with material behaviour beyond the elastic range, however they are not sensitive to cyclic load effects. Therefore the thermodynamic basics of inelastic material with fatigue damage parameter is here recovered.

Generally, damage parameter **d** is a fourth order tensor (Cauvin & Testa, 1999). If we assume fully isotropic material behaviour the damage tensor **d** is reduced to a scalar value d. In spite of the simplification of the nature of fatigue fracture, which is an anisotropic phenomenon, many authors apply isotropic damage parameter (Luccioni et. al., 1996; Lemaitre, 1992).

In this paper an constitutive model of an inelastic material with isotropic fatigue damage parameter is introduced. The free Helmholtz energy Ψ can be divided into two independent parts: corresponding to the elastic material behaviour Ψ^e and another corresponding to the plastic structure deformation Ψ^p.

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$$\Psi\left(\boldsymbol{\varepsilon}_{ij}{}^{e},\alpha,\beta\right)=\Psi^{e}\left(\boldsymbol{\varepsilon}_{ij}{}^{e},\beta\right)+\Psi^{p}(\alpha), \qquad (1)$$

where $\varepsilon_{ij}{}^{e}$ is the elastic part of the strain tensor and α, β are the internal material parameters corresponding to plasticity and elasticity.

The elastic part of the free energy, considering infinite small deformations and constant temperature, can be formulated as

$$\Psi^{e}\left(\varepsilon_{ij}^{e},\beta\right)=\frac{1}{2\rho}\left[\varepsilon_{ij}^{e}C_{ijkl}^{d}(\beta)\varepsilon_{kl}^{e}\right], \qquad (2)$$

where ρ is the material density, C^{d}_{ijkl} is the fourth order constitutive tensor which takes into account the evolution of the internal parameter β (Lemaitre, 1992; Chaboche, 1987). Generally

$$C_{ijkl}^{d}=g(\beta)C_{ijkl}^{0}, \qquad (3)$$

C^{0}_{ijkl} is the constitutive fourth order tensor without influence of internal parameter β, $g(\beta)$ is the scalar transformation function (whose argument is, generally, a tensor) of constitutive pure material tensor C^{0}_{ijkl}. Kachanov (1958), was the first, connected transformation function $g(\cdot)$ with the degradation of the material ($\beta\equiv d$). He took advantage of empirical linear dependence

$$g(d)=D-d \Rightarrow C_{ijkl}^{d}=(D-d)C_{ijkl}^{0}, \qquad (4)$$

where D is the critical damage value and d describes the level of degradation of the pure material. If $d=0$ the material is not damaged, and when $d=D$ material strength is collapsed. Usually parameter D is taken as 1, in spite of fact that scientific researches indicate that collapse of element may occur with much smaller value of parameter D (Szala, 1998). In this paper it is assumed that $D=1$.

After substitution of equation (4) into equation (2) we obtain the formula of elastic part of the free Helmholtz energy with internal damage parameter

$$\Psi^{e}=(1-d)\Psi^{e0}=\frac{1-d}{2\rho}\left[\varepsilon_{ij}^{e}C_{ijkl}^{0}\varepsilon_{kl}^{e}\right], \qquad (5)$$

where Ψ^{e0} is the elastic part of free energy without damage.

The Clausius – Duhem inequality in energetic form can be formulated as follows

$$\Gamma=m\left(-\dot{\Psi}-\eta\dot{\theta}\right)+\sigma_{ij}\dot{\varepsilon}_{ij}-\frac{1}{\theta}q_{i}\nabla\theta\geq 0. \qquad (6)$$

where (Sosnowski, 2003; Luccioni et al., 1996)

$$\sigma_{ij}=\rho\frac{\partial\Psi^{e}}{\partial\varepsilon_{ij}^{e}} \quad \text{and} \quad \eta=-\frac{\partial\Psi}{\partial\theta}, \qquad (7)$$

where η is the entropy , θ is the temperature and q_i is the vector of heat flux. The material derivative of the free Helmholtz energy (derivation of equations (1) and (2)) is as follows

$$\dot{\Psi}=\frac{\partial\Psi^{e}}{\partial\varepsilon_{ij}^{e}}\dot{\varepsilon}_{ij}^{e}+\frac{\partial\Psi^{e}}{\partial\beta}\dot{\beta}+\frac{\partial\Psi^{p}}{\partial\alpha}\dot{\alpha}. \qquad (8)$$

After substitution of equation (8) into equation (6) and making an assumption that $\beta\equiv d$ gives

$$\sigma_{ij}\dot{\varepsilon}_{ij}^{e}-\rho\frac{\partial\Psi^{e}}{\partial\varepsilon_{ij}^{e}}\dot{\varepsilon}_{ij}^{e}-\rho\frac{\partial\Psi^{e}}{\partial d}\dot{d}+\sigma_{ij}\dot{\varepsilon}_{ij}^{p}-,$$
$$\rho\frac{\partial\Psi^{p}}{\partial\alpha}\dot{\alpha}-\rho\eta\dot{\theta}-\frac{1}{\theta}q_{i}\nabla\theta\geq 0. \qquad (9)$$

The stress value in the damaged element can be calculated by a substitution of derivatives equation (5) into equation (7)

$$\sigma_{ij}=\rho\frac{\partial\Psi^{e}}{\partial\varepsilon_{ij}^{e}}=(1-d)C_{ijkl}^{0}(\varepsilon_{kl}-\varepsilon_{kl}^{p}),$$
$$\varepsilon_{ij}=\varepsilon_{ij}^{e}+\varepsilon_{ij}^{p} \qquad (10a)$$

or

$$\sigma_{ij}=(1-d)\sigma_{ij}^{o}, \qquad (10b)$$

where σ_{ij}^{o} is the stress in an undamaged element.

The second thermodynamical law indicates, that inequality (6) must be satisfied. Ostrowska-Maciejewska (1994) made an assumption, that inequality (6) must be satisfy separately for mechanical and thermal parts. In that case inequality (6), which describes energy dissipation, can be divided into two parts: the mechanical one (plastic and damage dissipation) and the thermal one (thermal dissipation):

$$\Gamma_{m}=\sigma_{ij}\dot{\varepsilon}_{ij}^{p}-\rho\frac{\partial\Psi^{p}}{\partial\alpha}\dot{\alpha}-\rho\frac{\partial\Psi^{e}}{\partial d}\dot{d}\geq 0$$
$$\Gamma_{\theta}=\frac{1}{\theta}q_{i}\nabla\theta\leq 0. \qquad (11)$$

The process of fatigue damage evolution must satisfy conditions in equation (11). In the fatigue analysis of structures two kinds of energy dissipation have to be taken into account: plastic dissipation and dissipation which corresponds with fatigue degradation of material. When a change of temperature has influence in the process also thermal dissipation must be taken into account.

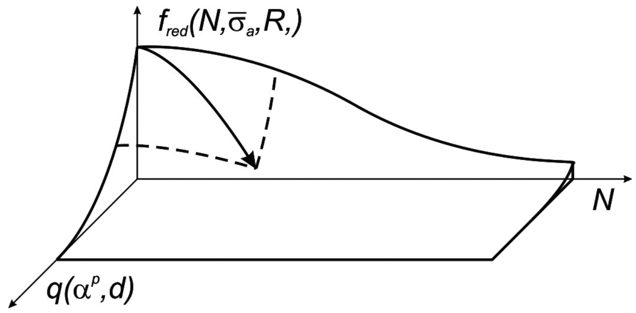


Fig. 1. Yield – damage evolution surface.

3. PLASTIC FUNCTION AND EVOLUTION OF DAMAGE IN FATIGUE ANALYSIS

Elastic limit is described by the yield criterion

$$F(\sigma_{ij},\alpha) = f(\sigma_{ij}) - K(\sigma_{ij},\alpha) \le 0 \qquad (12)$$

where $K(\sigma_{ij},\alpha)$ is yield limit.

The theory of plasticity establish that there exists a plastic potential. The scalar function $G(\boldsymbol{\sigma})$ is the plastic potential for $\dot{\boldsymbol{\varepsilon}}^p$. The flow law is (Sosnowski, 2003; Ostrowska-Maciejewska, 1994)

$$\varepsilon_{ij}^p = \lambda \frac{\partial G}{\partial \sigma_{ij}}, \quad \text{where } \lambda > 0, \qquad (13)$$

where λ is the scalar function. If we assume, that plastic potential G is identified with the flow function f, equation (13) will be the associated flow law

$$\dot{\varepsilon}_{ij}^p = \lambda \frac{\partial f}{\partial \sigma_{ij}}, \quad \text{where } \lambda > 0. \qquad (14)$$

In an analogy to the associated flow law, a material fatigue strength limit can be build.

$$F^D(\sigma_{ij},d) = \bar{\sigma}(\sigma_{ij}) - f_r(\sigma_{ij},d) f_{red}(N,\bar{\sigma}_a,R) \le 0, \qquad (15)$$

where $\bar{\sigma}$ is the equivalent stress (i.e. Huber – von Mises stress) in the damaged element, f_r is the fatigue endurance limit and f_{red} is the reduction of material strength function. Reduction of the material strength function f_{red} depends on the stress history (number of cycles loading – unloading), amplitude of equivalent stress $\bar{\sigma}_a$ and the stress ratio R. Similarly like in the case of theory of plasticity, an evolution of damage parameter is given by equation (Luccioni et al., 1996; Lemaitre, 1992; Oller et al., 2005)

$$\dot{d} = \mu \frac{\partial F^D}{\partial \bar{\sigma}}, \quad \dot{d} \ge 0 \qquad (16)$$

where μ has similar properties to λ.

The described above mechanical and damage processes allows to define a fatigue limit surface. The fatigue limit surface, presented in figure 1, has taken into account the plastic effects of deformation. The process of degradation of material strength is limited by the fatigue limit surface and it allows to simulate propagation of fatigue damage during the load history.

4. REDUCTION OF MATERIAL STRENGTH FUNCTION

An analytical form of the fatigue strength function with respect of the number of cycles N and the stress ratio R is proposed. This work extends the work of Luccioni et al. (1996), Chaboche (1987) and Oller et al. (2005).

$$SN(R,N) = S^{th}(R) + \left[S^u - S^{th}(R)\right] 10^{\left[-\alpha_t(R)\log_{10}(N)^{\beta}\right]}, \qquad (17)$$

where, for $|R| \le 1$

$$S^{th}(R) = S^e + (S^u - S^e)(0.5 + 0.5R)^{\gamma}$$

$$\alpha_t(R) = \alpha + (0.5 + 0.5R)\delta \qquad (18a)$$

and for $|R| > 1$

$$S^{th}(R) = S^e + (S^u - S^e)(0.5 + \frac{1}{2R})^{\gamma}$$

$$\alpha_t(R) = \alpha + (0.5 + \frac{1}{2R})\delta. \qquad (18b)$$

S^u is the material tensile strength, $\alpha, \beta, \gamma, \delta$ are the material parameters, S^{th} is the fatigue threshold function depending on the endurance fatigue stress S^e and the stress ratio R.

A modification of the S – N function formulas developed by Oller et al. (2005) is done in order to lead to a full agreement of the presented method with the classical stress methods. The introduced correction also takes into account an influence of mean stress value σ_{med} and the stress ratio R on fatigue life of the structure in agreement with Goodman's concept (Marczewska et al., 2006).

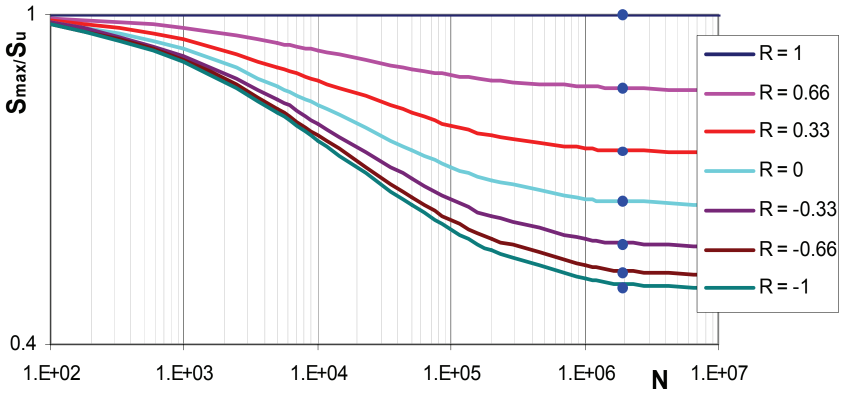


***Fig. 2.** The proposed S–N curve for different R values.*

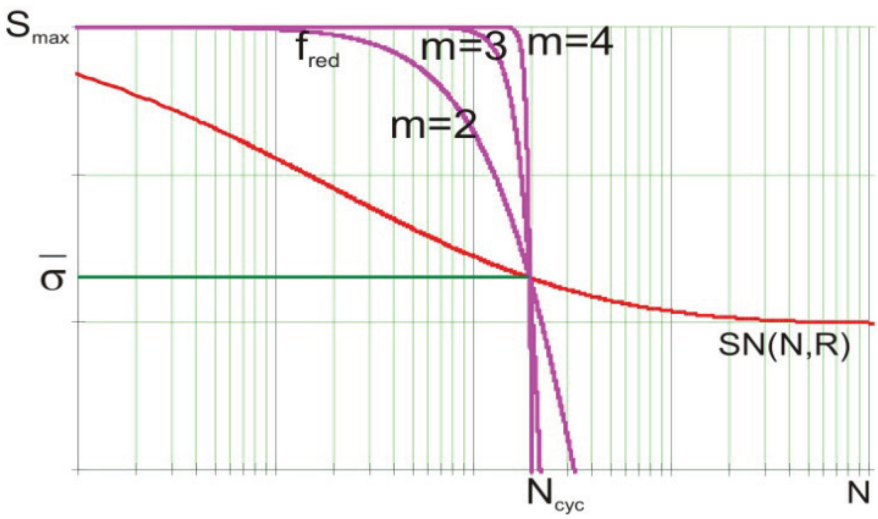


***Fig. 3.** Proposed strength reduction function.*

In figure 2 the proposed S – N function is presented. The influence of the stress ratio *R* is shown. The bold points indicate fatigue threshold values calculated by Goodman's concept. After transformation of equation (17) and substitution of the stress analysis result (i.e. FEM analysis introduced by Marczewska et al. (2006)) as $\overline{\sigma} \equiv S(N,R)$, we obtain the fatigue life of the structure. Fatigue life is expressed as a number of cycles to failure.

$$N_{cyc}(R) = 10^{\left[\left(\frac{\log\left(\frac{S^u - S^{th}(R)}{\overline{\sigma} - S^{th}(R)}\right)}{\alpha_t(R)}\right)^{\frac{1}{\beta}}\right]} \qquad (19)$$

The S – N curve determined by equation (17) and equation (19) allows to determine fatigue life of the structure where the load is characterized by constant amplitude and constant stress ratio. It is observed that in fatigue crack propagation consecutive material points are damaged (separated) along a crack. In order to describe evolution of fatigue crack authors introduce a material strength reduction function, $f_{red}(\overline{\sigma},R,N) \equiv 1-d$ (Luccioni et al., 1996; Oller et al., 2005).

$$f_{red}(\overline{\sigma},R,N) = 10^{\frac{\log\left(\frac{\overline{\sigma}}{S^u}\right)}{\log(N_{cyc})^{\beta m}}\log(N)^{\beta m}} \qquad (20)$$

where exponent *m* is the material brittleness measure. In figure 3 reduction of material strength function for different parameters *m* is presented.

The strength reduction function is identified with damage of material point $f_{red}(\overline{\sigma},R,N) \equiv 1-d$, in that case equations (4) and (10) can be re-written as

$$C^d_{ijkl} = f_{red}(\overline{\sigma},R,N)C^0_{ijkl} \quad \text{and}$$

$$\sigma_{ij} = f_{red}(\overline{\sigma},R,N)\sigma^o_{ij}. \qquad (21)$$

Furthermore

$$S^{ud} = f_{red}(\overline{\sigma},R,N)S^u, \qquad (22)$$

where S^{ud} is the damaged material strength.

5. NUMERICAL EXAMPLE – DOUBLE DISK ROTOR SHAFT SYSTEM

In the first step numerical calculations have been performed for the single-span rotor-shaft system with two heavy identical rotors of 2 kN of weight each located on the shaft between two journal bearings, as shown in figure 4. The total length of the shaft system is 2.315 m. The bearing span is equal to 1.6 m and the total weight of the considered rotor-shaft system is 5.9 kN. In this system the transverse crack of depth ratio a/ϕ=0.1 is assumed in the shaft cross-section corresponding to the one-half of the bearing span, as shown in figure 4, where ϕ=0.12 m.

The system dynamic response has been obtained for the range of system exploitation rotational speeds 500÷12000 rpm. Dynamic analysis of cracked shaft was performed by continous-discrete one-dimensional hybrid model approach developed by Szolc et al. (2006).

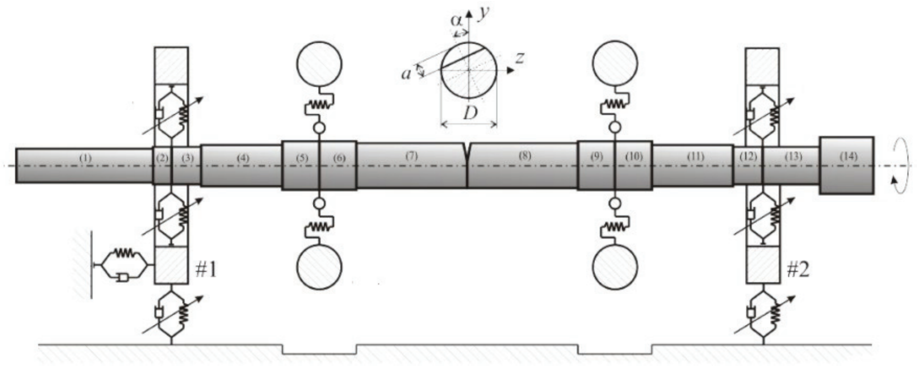


Fig. 4. Model of the two–bearing cracked rotor–shaft system (Szolc et. al., 2006).

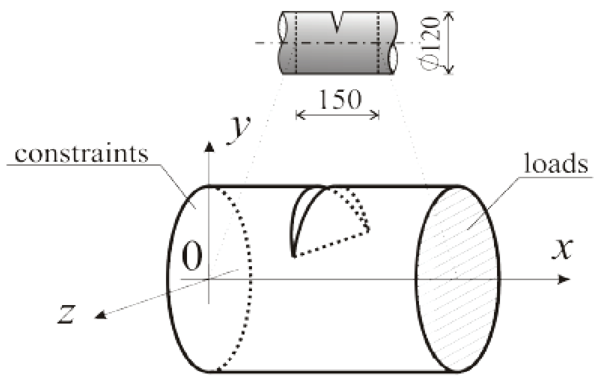


Fig. 5. Local model of the cracked shaft segment (Szolc et al., 2006).

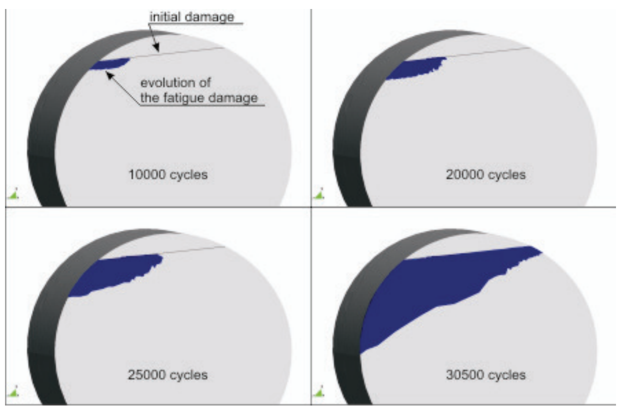


Fig. 6. Evolution of the fatigue damage.

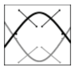
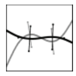
The quantities of particular interest in these investigations are the lateral vibration displacements of the shaft at the heavy rotor locations, bending and torsional moments as well as the longitudinal force transmitted by the shaft cracked zone.

The bending, torsional and axial loads in the form of time histories or amplitudes of dynamic forces and moments, obtained by means the one-dimensional hybrid model, are then included into the three-dimensional FEM model to act here as quasi-static excitations. Thus, the left-hand edge of the shaft segment shown in figure 5 has been assumed clamped and the right-hand edge is loaded by the bending moments acting in the 0xy and 0xz planes, the torsional moment acting around the 0x axis, as well as by the axial force acting along the 0x axis. Such an approach enables us to determine the time- and space-varying stress distribution in the cracked shaft segment. This stress distribution is regarded here as an input data for fatigue life estimation for the considered rotor-shaft as well as for prediction of the crack propagation process. This task is solved using presented fatigue-damage mechanics concept incorporated into the finite element code FEAP developed by Taylor (1999) and supplemented by the Marczewska et al. (2006). The material parameters are as follows: Young's modulus E = 210 GPa and Poison's ratio v = 0.3, the saturation flow stress corresponding to the material ultimate stress defining its static strength S^u = 520 MPa and the threshold stress is equal to S^{th} = 260 MPa.

The considered rotational speeds of the shaft (see table 1) correspond to the resonance frequencies of the rotor-shaft system, except 6000 rpm which is an operational speed. The fatigue analysis of the rotor-shaft system was carried out on the local model of the faulty shaft segment presented in figure 5. The local model is a short slice of the faulty shaft and it is modelled using 3D hexahedral finite elements. The left-hand side end of the faulty shaft segment was fully constrained. On the opposite right-hand side of this segment end the external dynamic load was imposed (see table 1).

The fatigue life of the shaft system was calculated by equation (19). In the case of the first three rotational speeds (3972 rpm, 6000 rpm and 7944 rpm) this kind of a crack does not involve prematural fatigue collapse. The stress level in the crack tip is below the material fatigue limit. The rotational speed 9990 rpm is a critical value for this kind of crack. A time of its duration is very short and it amounts about 3 min. The rotational speed 9990 rpm significantly exceeds the nominal operational speed of this rotor-shaft system. Additionally, then this rotor-shaft system would operate under resonant conditions.

Table 1. *Vibration amplitudes of inertial forces in cracked rotor-shaft system (Szolc et. al., 2006).*

rotational speed	3972 rpm	6000 rpm	7944 rpm	9990 rpm
form of the eigenvector	 bending form	design speed	 bending form	 torsional form
ampl. of the bending moment in vertical plane	100 [Nm]	220 [Nm]	340 [Nm]	740 [Nm]
ampl. of the bending moment in horizontal plane	95 [Nm]	210 [Nm]	370 [Nm]	730 [Nm]
ampl. of the torsional moment	100 [Nm]	13 [Nm]	200 [Nm]	11.5 [Nm]
ampl. of the longitudinal force	50 [N]	145 [N]	205 [N]	590 [N]
fatigue life	below fatigue limit	below fatigue limit	below fatigue limit	30999 rot., ~3 min of work

A simulation of fatigue crack propagation was also performed. The consecutive phases of crack increment are presented in figure 6. In the initial time period of operation, i.e. between 0 and 20 000 cycles, the speed of crack propagation is relatively small. A further operation leads to an increase of the crack propagation speed. After the overflow value of 30 500 cycles the crack grows unstably.

6. CONCLUSIONS

- A fatigue model based on continuum mechanic with its general expressions for the elasto-plastic-damage constitutive equations, sensitive for cyclic loads, has been presented.
- Damage mechanics in fatigue analysis allows to take into account interactions between elastic and plastic material behaviour.
- Effective numerical tool based on the FEM program to fatigue analysis of structures was developed.
- Knowledge about localization, direction and propagation speed of fatigue crack allows to improve the structure.
- It is possible to extend presented constitutive equations by thermal fatigue.

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SPRĘŻYSTO-PLASTYCZNY MODEL MATERIAŁU Z USZKODZENIEM W ANALIZIE ZMĘCZENIOWEJ METALI

Streszczenie

W pracy zaproponowano metodę oceny trwałości zmęczeniowej konstrukcji przy pomocy metody elementów skończonych oraz sprężysto-plastycznego modelu materiału z izotropowym parametrem uszkodzenia. Sformułowano prawa konstytutywne dla materiału na podstawie teorii plastyczności i mechaniki uszkodzeń wykorzystując wyniki prac Chaboche'a (1987), Luccioni'ego i innych (1996), Ollera i innych (2005) oraz własne, uproszczone modele obliczeniowe (Marczewska et al., 2006). Zaproponowano analityczną postać krzywej zmęczeniowej dla stali z uwzględnieniem współczynnika asymetrii cyklu obciążenia oraz funkcję degradacji materiału związaną ze zmęczeniem. Parametry materiałowe dopasowano do rzeczywistych badań eksperymentalnych. Opracowany algorytm obliczeń wykorzystano do symulacji procesu pękania zmęczeniowego wału maszyny wirnikowej.

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