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ACCURACY OF THE FINITE ELEMENT SOLUTION TO STEADY CONVECTION-DIFFUSION HEAT TRANSPORT EQUATION IN CONTINUOUS CASTING PROBLEM

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Abstract

Steady convection-diffusion equation plays an important role in description of the heat transfer in many technical problems. Finite element method is widely used to solve such problems. However, in convection dominated problems oscillatory solutions to the temperature field have been observed. Several methods have been proposed to overcome these difficulties. Transient solutions give satisfactory results, however the computational time is high and in the case of three dimensional problems difficult to accept. The problem is much more complicated because good looking results may give substantial error in the temperature field determination. In the paper the accuracy of several finite element solutions to the heat transfer in the continues steel casting problem has been discussed. The heat balance in the control volume has been computed in order to assess the solutions accuracy. New variational formulation has been proposed to solve steady convection-diffusion equation. The method uses Hermitian shape function and second order derivatives to the temperature field.

Key words: finite element method, heat balance, convective heat transfer

1. INTRODUCTION

Heat transfer in the continuous steel casting process is very difficult to model. It is due to interaction of fluid flow and heat conduction. The problem can be treated as a steady state since there is no temperature change in time and can be well described by the convection-diffusion heat transfer equation. The finite element method is most widely used to compute the temperature field in many technical problems. However, in the case of convection dominated processes oscillatory solution are observed and convergence is not obvious. Several methods have been proposed to overcome this difficulties. One of the most popular formulation uses the non symmetric weighting functions instead of the linear shape functions. This formulation was first proposed in [1]. The method is efficient but in some cases solutions to the temperature field are not satisfactory [3]. Transient or iterative methods can be successfully employed but the computation time is generally high in the case of three dimensional problems. Detailed description of the numerical methods can be found in [7]. The internal heat source in the convection-diffusion heat transport equation makes the solution much more complicated. This issue has been addressed in work [4] and [5]. However, stable and good looking results may not be accurate. It makes the interpretation of the temperature computation results very difficult. It is not possible to validate the

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temperature field each time by the measurement. In the paper a method of validation based on the heat balance has been proposed. Accuracy of the solutions for a standard and modified Galerkin method as well as for a variation formulation have been presented.

2. MATHEMATICAL MODELS

In the continuous steel casting process the heat transfer is mainly based on conduction but fluid flow in the liquid region affects the temperature field significantly. Forced and free convection contribute to the heat transfer. Four methods of solving the heat transfer problem have been tested. The temperature field *T(x,y,z)* for the steady-state heat conduction can be calculated from the equation:

$$\int_V \left[\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+q_V - \varrho c\left(v_x\frac{\partial T}{\partial x}v_y\frac{\partial T}{\partial y}v_z\frac{\partial T}{\partial z}\right)\right]dV=0 \tag{1}$$

where: T – temperature, K; v_x, v_y, v_z – velocity field, m/s; λ – thermal conductivity, W/(m·K); q_v – internal heat source, W/m^3; c – specific heat, J/(kg·K); ρ – density, kg/m^3.

The Galerkin method [7,2] is the most popular solution to Eq (1) and yields the set of linear equations:

$$(K_{nn}+W_{nn})P_n = G_n \tag{2}$$

where: n is number of unknowns and P_n are unknown parameters. Several formulations depending on the choice of the weighting and shape functions can be developed to define the matrices K_{nn}, W_{nn} and vector G_n

Method 1

The first method is a standard Galerkin solution to equation (1). The domain V is divided into 8 node elements with linear weighting and shape functions [2]: It gives for one element

$$W_{ij}=\sum_{k=1}^{8}\varrho^k c^k\left(v_x^k N_i\frac{\partial N_j}{\partial x}+v_y^k N_i\frac{\partial N_j}{\partial y}+v_z^k N_i\frac{\partial N_j}{\partial z}\right)D_V^k \tag{3}$$

$$K_{ij}=\sum_{k=1}^{8}\lambda^k\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}+\frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\right)D_V^k+\sum_{s=1}^{8}L^s\sum_{k=1}^{4}N_iN_j\propto^k D_s^k \tag{4}$$

$$G_i=\sum_{k=1}^{8}q_V^k N_i D_V^k\sum_{s=1}^{6}L^s\sum_{k=1}^{4}N_i\left(\propto^k T_a^k - q^k\right)D_s^k \tag{5}$$

$$i=1,\dots,8;\quad j=1,\dots,8$$

where: N_i are linear shape functions, T_a – ambient temperature, K; q – heat flux rate, W/m^2; α – heat transfer coefficient, W/(m^2·K); k – Gaussian integration point number, D_V – constant resulting from Gaussian quadrature formula and volume element coordinates transformation to a prism region, D_s – constant resulting from Gaussian quadrature formula and surface element coordinates transformation to a square region, L^s – constant equal 0 or 1 if on element surface boundary conditions are specified. The unknown parameters P_i in equation (2) resulting from this formulation are temperature values in element nodes. Gaussian elimination scheme is used to solve band diagonal system of linear equations (2).

Method 2

The second method is similar to the first method. The domain V is also divided into 8 node elements. However, instead of the linear weighting and shape functions the Hermitian interpolation functions H_i are employed. It leads to:

$$W_{ij}=\sum_{k=1}^{27}\varrho^k c^k\left(v_x^k H_i\frac{\partial H_j}{\partial x}+v_y^k H_i\frac{\partial H_j}{\partial y}+v_z^k H_i\frac{\partial H_j}{\partial z}\right)D_V^k \tag{6}$$

$$K_{ij}=\sum_{k=1}^{27}\lambda^k\left(\frac{\partial H_i}{\partial x}\frac{\partial H_j}{\partial x}+\frac{\partial H_i}{\partial y}\frac{\partial H_j}{\partial y}+\frac{\partial H_i}{\partial z}\frac{\partial H_j}{\partial z}\right)D_V^k+\sum_{s=1}^{8}L^s\sum_{k=1}^{9}N_iN_j\propto^k D_s^k \tag{7}$$

$$G_i = \sum_{k=1}^{27} q_V^k H_i D_V^k + \sum_{s=1}^{6} L^s \sum_{k=1}^{9} H_i\left(\propto^k T_a^k - q^k\right) D_s^k \quad (8)$$

$$i = 1, \dots, 64 \, ; \quad j = 1, \dots, 64$$

To give an example, for the first node with the set of local coordinates: $\eta_1 = 0; \ \eta_2 = 0; \ \eta_3 = 0$ we can write:

$$\begin{aligned}
&T(\eta_1,\eta_2,\eta_3) = P_1 \, ; \quad H_1 = G_1^0(\eta_1)G_1^0(\eta_2)G_1^0(\eta_3) \\
&\frac{\partial T(\eta_1,\eta_2,\eta_3)}{\partial \eta_1} = P_2 \, ; \quad H_2 = G_1^1(\eta_1)G_1^0(\eta_2)G_1^0(\eta_3) \\
&\frac{\partial T(\eta_1,\eta_2,\eta_3)}{\partial \eta_2} = P_3 \, ; \quad H_3 = G_1^0(\eta_1)G_1^1(\eta_2)G_1^0(\eta_3) \\
&\frac{\partial T(\eta_1,\eta_2,\eta_3)}{\partial \eta_3} = P_4 \, ; \quad H_4 = G_1^0(\eta_1)G_1^0(\eta_2)G_1^1(\eta_3) \\
&\frac{\partial^2 T(\eta_1,\eta_2,\eta_3)}{\partial \eta_1 \partial \eta_2} = P_5 \, ; \quad H_5 = G_1^1(\eta_1)G_1^1(\eta_2)G_1^0(\eta_3) \\
&\frac{\partial^2 T(\eta_1,\eta_2,\eta_3)}{\partial \eta_2 \partial \eta_3} = P_6 \, ; \quad H_6 = G_1^0(\eta_1)G_1^1(\eta_2)G_1^1(\eta_3) \\
&\frac{\partial^2 T(\eta_1,\eta_2,\eta_3)}{\partial \eta_1 \partial \eta_3} = P_7 \, ; \quad H_7 = G_1^1(\eta_1)G_1^0(\eta_2)G_1^1(\eta_3) \\
&\frac{\partial^3 T(\eta_1,\eta_2,\eta_3)}{\partial \eta_1 \partial \eta_2 \partial \eta_3} = P_8 \, ; \quad H_8 = G_1^1(\eta_1)G_1^1(\eta_2)G_1^1(\eta_3)
\end{aligned} \quad (9)$$

where:

$$G_1^0 = 1 - 3\eta_n^2 + 2\eta_n^3$$
$$G_1^1 = \eta_n - 2\eta_n^2 + \eta_n^3 \quad n = 1,2,3 \quad 0 \le \eta \le 1 \quad (10)$$

The unknown parameters P_i in equation (2) in the case of this method are temperatures and their derivatives in element nodes. Gaussian elimination scheme is used to solve band diagonal system of linear equations (2).

Method 3

Direct variational methods can be also used to solve heat transport equation. However, the interpolation functions must give the possibility to calculate first and second derivatives to the temperature field. In order to fulfill this requirements the Hermitian interpolation functions are employed to approximate temperature *T(x,y,z)* and its derivatives in the Eq. (1). Simple derivation of the Hermitian approximation functions lids to the error norm, which for one element takes the form:

$$\Psi = \sum_{k=1}^{27}\left[\lambda^k\left(\sum_{i=1}^{64} P_i \frac{\partial^2 H_i}{\partial x^2} + \sum_{i=1}^{64} P_i \frac{\partial^2 H_i}{\partial y^2} + \sum_{i=1}^{64} P_i \frac{\partial^2 H_i}{\partial z^2}\right) + q_V^k - \varrho^k c^k \left(v_x^k \sum_{i=1}^{64} P_i \frac{\partial H_i}{\partial x} + v_y^k \sum_{i=1}^{64} P_i \frac{\partial H_i}{\partial y} + v_z^k \sum_{i=1}^{64} P_i \frac{\partial H_i}{\partial z}\right)\right]^2 D_V^k \quad (11)$$

The minimization of the error norm gives parameters P_i which represent temperatures and their derivatives in element nodes. Alignment of P_i parameters for the first node is given by equation (9). The variable metric method based on BFGS algorithm [6] is used to minimize the error norm Ψ.

Method 4

Implementation of the Hermitian interpolation functions and volume element coordinates transformation to a prism region and the use of Gaussian quadrature formula to the integrals of equation (1) have lead to another form of equation (1)

$$\Phi = \sum_{k=1}^{27}\left[\lambda^k\left(\sum_{i=1}^{64} P_i \frac{\partial^2 H_i}{\partial x^2} + \sum_{i=1}^{64} P_i \frac{\partial^2 H_i}{\partial y^2} + \sum_{i=1}^{64} P_i \frac{\partial^2 H_i}{\partial z^2}\right) + q_V^k - \varrho^k c^k \left(v_x^k \sum_{i=1}^{64} P_i \frac{\partial H_i}{\partial x} + v_y^k \sum_{i=1}^{64} P_i \frac{\partial H_i}{\partial y} + v_z^k \sum_{i=1}^{64} P_i \frac{\partial H_i}{\partial z}\right)\right] D_V^k \quad (12)$$

The unknown parameters P_i are temperatures and their derivatives in element nodes. The parameters P_i can be easily calculated from the set of linear equations:

$$\frac{\partial \Phi}{\partial P_i} = 0 \quad (13)$$

Gaussian elimination scheme is used to solve band diagonal system of linear equations (13).

3. STEADY ONE DIMENSIONAL HEAT CONDUCTION TEST

The methods of solution to equation (1) have been tested on one dimensional heat conduction test. One dimensional heat conduction in cuboid with the dimensions of: 8 mm × 80 mm × 800 mm, in the direction of coordinate system, respectively, has been considered. The boundary conditions are as follows: on the front surface *T*(*z*=0)=1500°C, on the rear surface *q*(*z*=800)= 10000 W/m^2, on the side surfaces *q*= 0. Steel thermal conductivity λ is 30 W/(m·K), steel density ρ is 7600 kg/m^3, steel specific heat *c* is 650 J/(kg·K). The control volume has been divided into 4 elements in each direction of the coordinate system. The results of computations have been presented in figure 1. The temperature drop is linear in the direction of z coordinate. It can be readily calculated that the slop of the temperature line is $-q/\lambda$. Methods 2, 3 and 4 have given the exact solution to one dimensional heat conduction test.

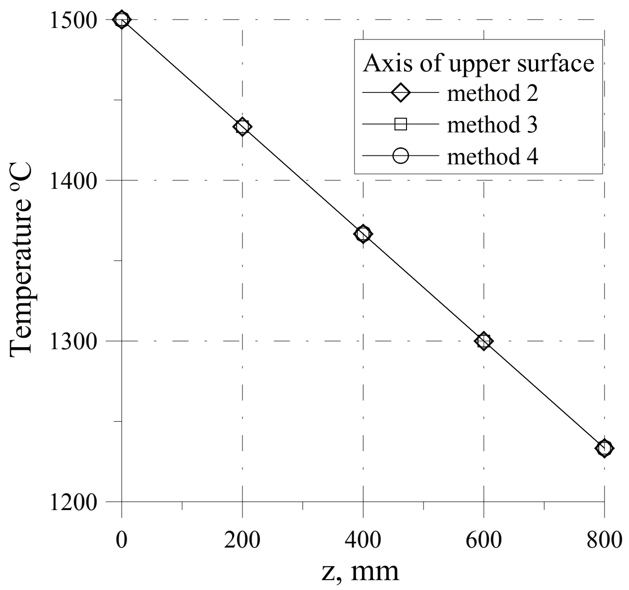

Fig. 1. *Variation of the temperature in z direction for one dimensional steady conduction test.*

4. TEMPERATURE MODELING IN THE MOULD REGION

The second set of computations has been performed for the test which models heat transfer in the mould region of the continuous steel caster. The cross section of the mould is 160 mm × 160 mm. The mould is 800 mm long. Steel flows with the constant velocity of 50 mm/s. The inlet temperature of steel is 1500°C. The mould temperature T_a is 100°C. Due to the symmetry of the flow only ¼ of the channel cross section has been modeled. The control volume has been divided into 4 elements in x and y direction. In the z direction of steel flow 8 elements have been employed. The computations have been performed for two sets of boundary conditions:

Case I – constant heat flux at a rate of q = 100000 W/m^2 at the mould surface.

Case II – variable heat flux at a rate of $q = \alpha(T - T_a)$ at the mould surface. The heat conduction coefficient α is 1000 W/(m^2·K).

The four methods described in chapter 2 have been employed to solve equation (1). In order to estimate the accuracy of the methods the heat balance in the control volume has been calculated. In addition to the heat balance, an integral of the difference between prescribed boundary conditions and the heat flux resulting from the computed temperature field has been calculated. This integral represents the error of the boundary conditions which have not been satisfied in full by the numerical methods.

The results of computation for the Case I are presented in figures 2, 3, 4 and 5. In figure 2 the temperature variations on the strip surface are presented. The four methods have given very low values of the integral of equation (1). The results of the integral of equation (1) are presented in figure 3. The maximum value is below 0.002 W. However, the temperature variations on the strip surface are significantly different. The standard and modified Galerkin methods (Method 1 and 2) have given very similar temperature drops along the strip surface. The error of the heat balance presented in figure 4 indicates that the Galerkin methods fulfilled the heat balance in the control volume. The variational solution (Method 3) has given very low error of the heat balance and boundary condition error. The temperature distribution on the strip surface calculated by the variational method varies slightly from that given by the Galerkin methods. The reason for that is probably in the representation of the boundary conditions. The error in the boundary conditions presented in figure 5 is the highest for the Galerkin method. The method 4 has given significant error in the heat balance and the temperature variation on the strip surface is oscillatory. The computations performed for the Case I have shown that the Galerkin methods and the variational solution have given very good results. The variational solution (Method 3) may be slightly more accurate.

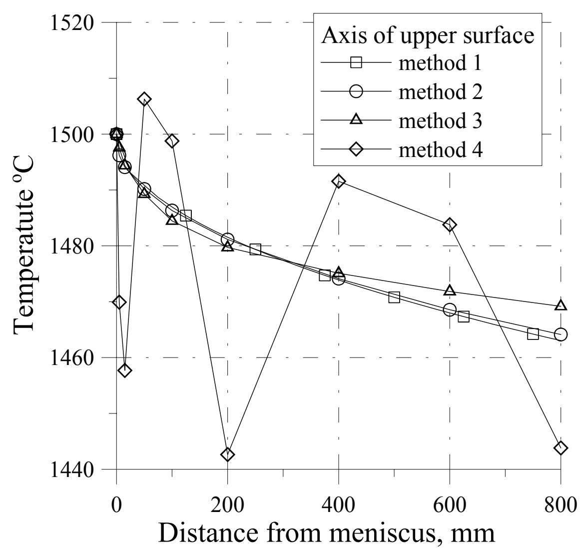

Fig. 2. *Temperature variation on the strip surface for the case I.*

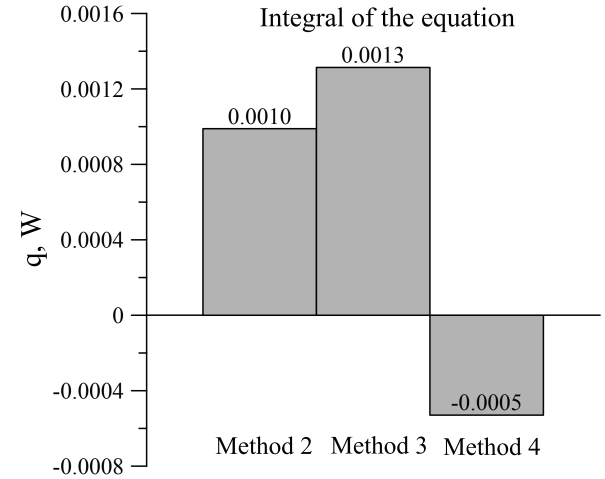

Fig. 3. The values of the Eq. 1 integral for the case I.

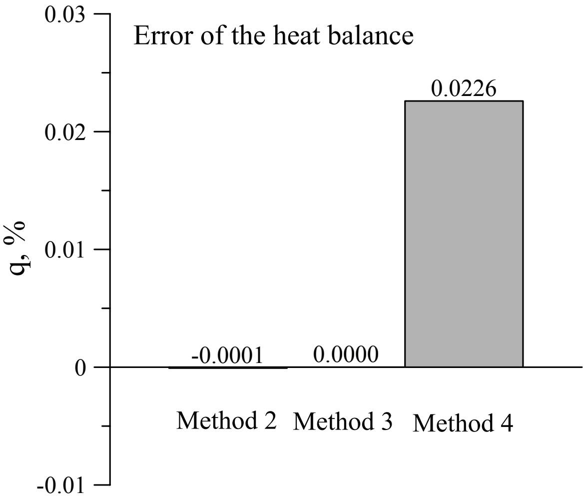

Fig. 4. The values of the heat balance error for the case I.

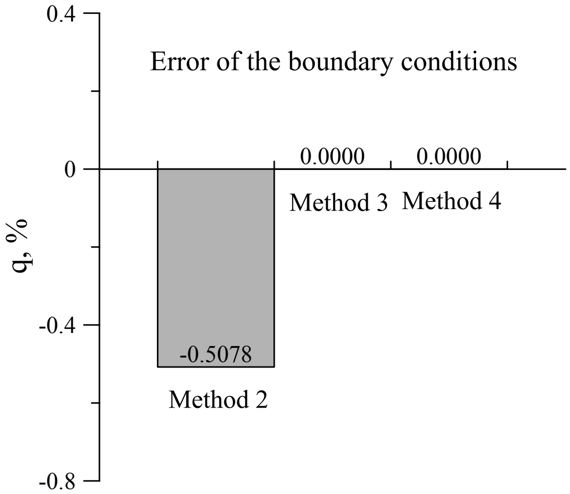

Fig. 5. The values of the boundary condition error for the case I.

The results of computation of the temperature field for the Case II have been presented in figures 6, 7, 8 and 9. The boundary conditions in this case are specified as functions of the strip surface temperature, the mould temperature and the heat transfer coefficient. Even for steady formulation the iterative solution must be employed in order to gradually improve the strip temperature value which has significant influence on the heat transfer rate on the strip surface. Tenth iteration have been used in order to stabilize the solution. The method 4 is not shown in figures since this method did not give any reasonable results. Similarly as in the Case I the Galerkin methods have given almost identical temperature variation on the strip surface, shown in figure 6. Values of the integral of equation (1) shown in figure 7 are low both for the Galerkin and the variational formulation. However, the strip surface temperature resulting from the variational solution is higher from that obtained with the Galerkin method. The reason for that may lie in the heat balance error, figure 8, and the boundary condition error, figure 9, which are essentially higher in the case of the Galerkin formulation.

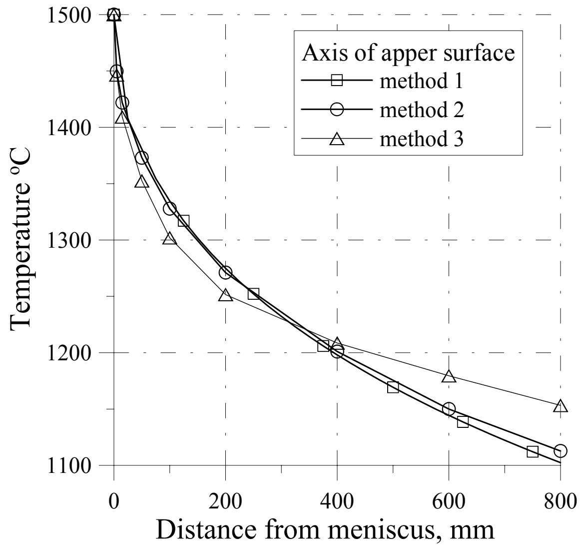

Fig. 6. Temperature variation on the strip surface for the case II.

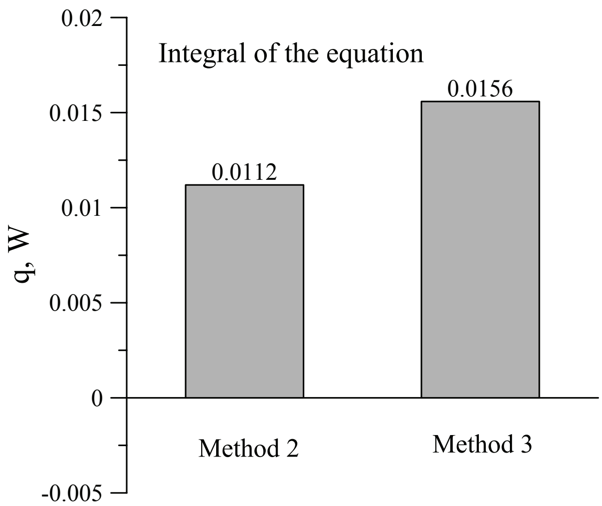

Fig. 7. The values of the Eq. 1 integral for the case II.

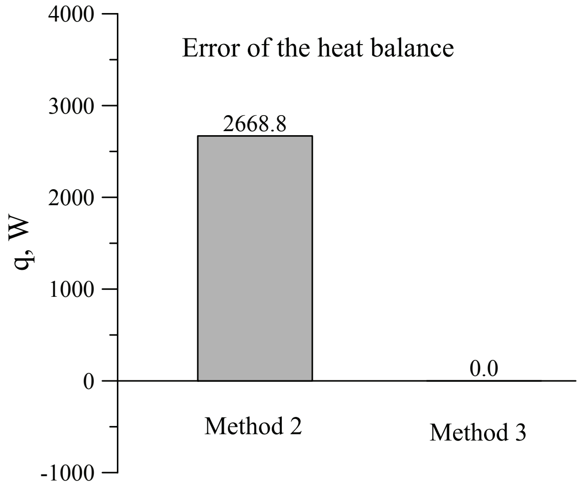

Fig. 8. *The values of the heat balance error for the case II.*

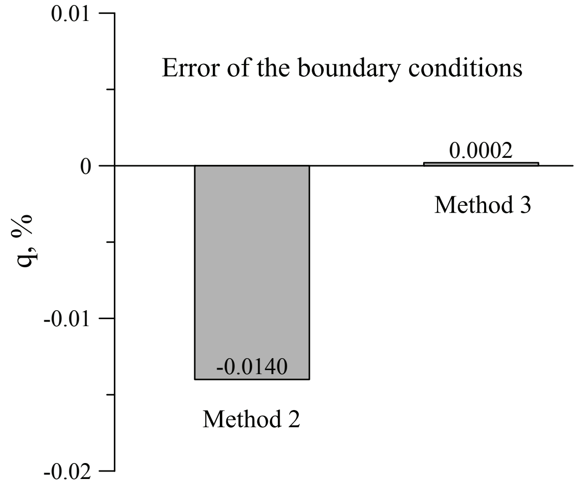

Fig. 9. *The values of the boundary condition error for the case II.*

5. CONCLUSIONS

The four methods of the solution to steady convection – diffusion heat transfer equation have been compared. The standard Galerkin method and the Galerkin method with the Hermitian weighting and shape function have given almost identical temperature fields for all the tests. The variational solution, which is based on the minimization of the heat transfer equation error, results in lover values of the heat balance error and the boundary condition error. The temperature changes on the strip surface predicted by the Galerkin and variational formulations are slightly different. It can be caused by the differences in the boundary condition handling in numerical procedures. Further work have to be done in order to explain large temperature oscillations encountered in the case of method 4 which leads to a linear set of equations, similar to that of the Galerkin method. The developed Galerkin formulation with the Hermitian weighting and shape function is very rapid and stable even for a control volume 10 m long and can be used to compute the temperature field in the hole continuous casting line very efficiently.

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REFERENCES

1. Heinrich, J.C., Huyakorn, P.S., Zienkiewicz, O.C., An upwind finite element scheme for two-dimensional convective transport equation, International Journal for Numerical Methods in Engineering, 11, 1977, 131-143.
2. Malinowski, Z., Numeryczne modele w przeróbce plastycznej i wymianie ciepła, Wydawnictwa AGH, Kraków, 2005 (in Polish).
3. Malinowski, Z., Stabilizacja algorytmu MES w rozwiązaniach zagadnienia przewodzenia ciepła w warunkach konwekcji ze źródłem ciepła, Informatyka w Technologii Materiałów, 1, 2001, 135-146 (in Polish).
4. Pao, C.V., Numerical Methods for Coupled System of Nonlinear Parabolic Boundary Value Problems, Journal of Mathematical Analysis and Applications, 151, 1990, 581-608.
5. Pao, C.V., Numerical Methods for Time-Periodic Solutions of Nonlinear Parabolic Boundary Value Problems, Journal of Numerical Analysis, 2, 2001, 647-667.
6. Press, W. H., Teukolsky, S. A., Vetterling, W. T., Flannery, B. P., Numerical Recipies in Fortran 77, Cambridge University Press, Cambridge, 1997.
7. Zienkiewicz, O.C., Taylor, R.L., Finite Element Method Fifth edition, Vol. 3: Fluid Dynamics, Butterworth-Heinemann, Oxford, 2000.

DOKŁADNOŚĆ ROZWIĄZANIA OPARTEGO O ELEMENTY SKOŃCZONE RÓWNANIA USTALONEGO STANU KONWEKCJA-DYFUZJA WYMIANY CIEPŁA W PROCESIE CIĄGŁEGO ODLEWANIA STALI

Streszczenie

Równanie ustalonego stanu konwekcja-dyfuzja odgrywa ważną rolę w opisie wymiany ciepła w wielu problemach technicznych. Wśród metod numerycznych, umożliwiających rozwiązanie powyższego równania, szerokie zastosowanie znalazła metoda elementów skończonych. Jednakże rozwiązanie równania w warunkach konwekcji w przypadku metody elementów skończonych nie zawsze pozwala uzyskać stabilne rozwiązania. Zaobserwowano wyniki oscylujące wokół temperatury powierzchni. Tego typu problemy próbowano pokonać poprzez zastosowanie odpowiedniej metody. Przejściowe rozwiązania pozwoliły uzyskać satysfakcjonujące wyniki, jednakże czas obliczeniowy okazał się długi i w przypadku trójwymiarowych zagadnień trudny do zaakceptowania. Jednym z istotnych problemów przy doborze odpowiedniej metody są poprawnie zinterpretowane wyniki obliczeń. Niejednokrotnie pozornie dobrze wyglądające wyniki pola temperatury są obarczone istotnym błędem.

W pracy oszacowano bilans cieplny w kontrolowanej objętości w procesie ciągłego odlewania stali. Przeprowadzono dyskusję oceny dokładności rozwiązania równania ustalonego stanu konwekcja-dyfuzja przy zastosowaniu wybranych metod opartych o elementy skończone. Zaproponowano nowe wariacyjne sformułowanie umożliwiające rozwiązanie powyższego równania. W metodzie zastosowano funkcje kształtu Hermite'a.

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