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NONLOCAL INTEGRAL FORMULATION FOR A PLASTICITY-INDUCED DAMAGE MODEL

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Abstract

Ductile failure prediction commonly involves the use of the finite element method within an elasto-plasticity framework coupled with damage. As strain localisation takes place, the partial differential equilibrium equation experiences a loss of ellipticity and the numerical result becomes mesh dependent. One possible approach is to regularise the solution by the introduction of an internal length, which is related to the microscopic structure of the material. In this context, nonlocal integral and gradient-enhanced formulations [2] are commonly employed as regularisation strategies. In this work, a finite element formulation based on a nonlocal integral implementation proposed by Strömberg & Ristinmaa [7] is extended for a simplified version of Lemaitre's ductile damage model [1]. Since only the computation of the internal force is affected in this strategy, its implementation in existing FE codes is relatively straightforward. The results of a sample case show that the implemented strategy acts as localisation limiter.

Key words: ductile failure, nonlocal models, damage, finite element method

1. INTRODUCTION

When inelastic structures experience critical loading conditions, it is often observed the appearance of a localised straining region after an initially smooth strain distribution. In particular, when micro-voids nucleate, grow and coalesce, failure onset takes place on the Representative Volume Element (RVE) followed by the appearance and propagation of a macroscopic crack, leading to the final failure of the structure.

The phenomenon of strain localisation clearly represents a difficulty for an accurate prediction of ductile failure. In many circumstances, the mathematical and numerical models that describe the softening behaviour, within the standard local continuum theory, cannot explain correctly the dissipation of energy of a failure process. They may lead to very low values of dissipated energy, tending to zero. This situation is associated to the loss of ellipticity of the equilibrium differential equation and it is reflected in the finite element numerical solution by a pathological sensitivity of the results upon mesh refinement.

Based on the work of Strömberg & Ristinmaa [7], we present herein a finite element formulation of a nonlocal integral scheme. A simplified version of Lemaitre's damage model is considered for the constitutive modelling of the ductile behaviour.

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2. FINITE ELEMENT NONLOCAL FORMULATION

The finite element discrete boundary value problem for *quasi*-static problems is formulated as

$$\mathbf{f}^{\text{int}}(\mathbf{u}) - \mathbf{f}^{\text{ext}} = \mathbf{0} \quad (1)$$

where $\mathbf{f}^{\text{int}}$ and $\mathbf{f}^{\text{ext}}$ are, respectively, the internal and external global force vectors. The problem consists in finding the global vector of nodal displacements, $\mathbf{u}$, such that equation (1) holds. This equation is then normally solved by an iterative/incremental strategy like the Newton-Raphson method. In this numerical scheme, the internal force vector calculation requires the computation of the stress tensor, $\boldsymbol{\sigma}$, at every material point of the structure. This procedure, usually called *stress update*, is often carried out by means of a *fully implicit elastic predictor/return mapping algorithm* [6], which is the strategy employed in the present work.

2.1. Damage constitutive model

Lemaitre [4,5] has proposed a set of constitutive equations for elasto-plasticity coupled with damage aiming to predict the behaviour of ductile metals in the presence of internal degradation. This model is based on the assumption of *strain equivalence* and on the concept of *effective stress*. Furthermore, Lemaitre's model includes the evolution of damage as internal variable as well as non-linear isotropic and kinematic hardening. Considering a uniaxial stress state σ, the effective stress is given by

$$\sigma_{\text{eff}} = \frac{\sigma}{1-D} \quad (2)$$

where D is the *damage* scalar variable with $D = 0$ corresponding to the virgin material and $D = 1$ representing the situation where a total loss of strength of the material is observed.

The simplified version of Lemaitre's damage model [1] considers the model with evolution of isotropic hardening only. This material model has proved effective in situations where kinematic hardening can be disregarded. Furthermore, as shown by de Souza Neto [1], its computational implementation can be relatively straightforward and the resulting algorithm is very efficient.

2.2. Nonlocal formulation

The first step to extend the simplified Lemaitre's damage model into its nonlocal counterpart is the definition of the nonlocal variable. In the present formulation, we select the isotropic hardening internal variable, R. Rewriting R as a nonlocal variable, we have

$$\bar{R}(\boldsymbol{x}) = \int_V \alpha(\boldsymbol{x},\boldsymbol{\xi})R(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} \quad (3)$$

where $\alpha(\boldsymbol{x},\boldsymbol{\xi})$ is a given nonlocal weight function. Near the boundary, the function must be rescaled in order to preserve uniform distributions. The most frequently used modification [3] is to set

$$\alpha(\boldsymbol{x},\boldsymbol{\xi}) = \frac{\alpha_0(\|\boldsymbol{x}-\boldsymbol{\xi}\|)}{\int_V \alpha_0(\|\boldsymbol{x}-\boldsymbol{\zeta}\|)\mathrm{d}\boldsymbol{\zeta}} \quad (4)$$

where $\alpha_0(r) = \alpha_0(\|\boldsymbol{x}-\boldsymbol{\xi}\|)$ is a monotonically decreasing nonnegative function of the distance between points $\boldsymbol{x}$ and $\boldsymbol{\xi}$, where $r = \|\boldsymbol{x}-\boldsymbol{\xi}\|$. This function can assume, for an instance, the form of a Gauss distribution, expressed by

$$\alpha_0(r) = \exp\left(-\frac{r^2}{2\ell^2}\right) \quad (5)$$

where ℓ is a material parameter called *internal length*.

Following a similar strategy used by Jirásek [3], we replace the integral of equation (3) by a numerical integration with a Gaussian quadrature as

$$\bar{R}_i = \sum_{j=1}^{n_{gpi}} w_j J_j \alpha_{ij} R_j \quad (6)$$

where $\bar{R}_i$ is the nonlocal isotropic hardening variable counterpart of the i^{th} global Gauss point, R_j is the isotropic hardening variable value of the global Gauss point j and n_{gpi} is the number of Gauss points which influence the point i (inside an interaction radius represented by ℓ). The parameters w_j and J_j are, respectively, the weight of the integration point and the Jacobian of the isoparametric transformation evaluated at point j. Thus, the integral is worked out over a domain composed by a finite number of elements. Finally, the parameter α_{ij} is given by

$$\alpha_{ij} = \frac{\alpha_0(\|\boldsymbol{x}_i-\boldsymbol{x}_j\|)}{\sum_{m=1}^{n_{gpi}} w_m J_m \alpha_0(\|\boldsymbol{x}_i-\boldsymbol{x}_m\|)}. \quad (7)$$

The evolution of the local isotropic hardening variable in the local constitutive model is given by

$$\dot{R} = \dot{\gamma}. \quad (8)$$

Using a backward Euler discretisation, we have

$$(R_{n+1} - R_n) = \Delta\gamma. \quad (9)$$

However, in the nonlocal model, the rate of the nonlocal isotropic hardening variable must be considered. Thus, its discretised form is written as

$$\bar{R}_{n+1} = \bar{R}_n + \sum_{j=1}^{n_{gpi}} w_j J_j \alpha_{ij} \Delta\gamma_j. \quad (10)$$

This clearly shows that the original residual equation (as shown by de Souza Neto [1]) is not locally valid anymore and it must be converted to a vector field. For the sake of simplicity, we omit the final vector residual equation limiting ourselves to write a general component i without loss of generality. Thus,

$$\bar{F}_i(\Delta\boldsymbol{\gamma}) = 3G_i\Delta\gamma_i - (1 - D_{i_n})\left[q_i^{trial} - \sigma_{y_i}(\bar{R}_i(\Delta\boldsymbol{\gamma}))\right] + \frac{\left[q_i^{trial} - \sigma_{y_i}(\bar{R}_i(\Delta\boldsymbol{\gamma}))\right]^2}{3G_i}\left(\frac{-Y_i(\Delta\boldsymbol{\gamma})}{r}\right)^s \quad (11)$$

where the elastic energy release rate is given by

$$-Y_i(\Delta\boldsymbol{\gamma}) = \frac{\left[\sigma_{y_i}(\bar{R}_i(\Delta\boldsymbol{\gamma}))\right]^2}{6G_i} + \frac{\tilde{p}_i^2}{2K_i}, \quad (12)$$

q^{trial} is the damaged von Mises equivalent stress at the elastic trial state, $\tilde{p}$ is the hydrostatic pressure, G is the shear modulus, K is the bulk modulus, σ_y is the yield stress in function of the nonlocal isotropic hardening variable and s and r are damage material parameters. The subscript $n+1$ was omitted for convenience. The vectorial residual equation $\bar{\boldsymbol{F}}(\Delta\boldsymbol{\gamma}) = \boldsymbol{0}$ must be solved properly in order to update the stress state at every material point. Due to the degree of nonlinearity of the equation, we use the Newton-Raphson method. Therefore, the problem is reduced to the solution of a system of equations as

$$\left(\left.\frac{\partial\bar{\boldsymbol{F}}}{\partial\Delta\boldsymbol{\gamma}}\right|_{\Delta\boldsymbol{\gamma}^k}\right)\delta\boldsymbol{\gamma} = -\bar{\boldsymbol{F}}(\Delta\boldsymbol{\gamma}^k) \quad (13)$$

where

$$\delta\boldsymbol{\gamma} = \Delta\boldsymbol{\gamma}^{k+1} - \Delta\boldsymbol{\gamma}^k \quad (14)$$

and $\partial\bar{\boldsymbol{F}}/\partial\Delta\boldsymbol{\gamma}$ is a $n \times n$ matrix given by

$$\frac{\partial\bar{\boldsymbol{F}}}{\partial\Delta\boldsymbol{\gamma}} = \begin{bmatrix} \frac{\partial\bar{F}_1}{\partial\Delta\gamma_1} & \frac{\partial\bar{F}_1}{\partial\Delta\gamma_2} & \cdots & \frac{\partial\bar{F}_1}{\partial\Delta\gamma_n} \\ \frac{\partial\bar{F}_2}{\partial\Delta\gamma_1} & \frac{\partial\bar{F}_2}{\partial\Delta\gamma_2} & \cdots & \frac{\partial\bar{F}_2}{\partial\Delta\gamma_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\bar{F}_n}{\partial\Delta\gamma_1} & \frac{\partial\bar{F}_n}{\partial\Delta\gamma_2} & \cdots & \frac{\partial\bar{F}_n}{\partial\Delta\gamma_n} \end{bmatrix} \quad (15)$$

where n is the global number of Gauss points of the body.

It is important to remark that $\partial\bar{\boldsymbol{F}}/\partial\Delta\boldsymbol{\gamma}$ is generally *unsymmetric* and that a diagonal matrix is obtained if the local case is considered. We can simplify this matrix by considering only the terms in the diagonal of equation (15). Thus, the solution at every point is given by

$$\Delta\gamma_i^{k+1} = \Delta\gamma_i^k - \frac{\bar{F}_i(\Delta\boldsymbol{\gamma})}{\bar{F}_i'(\Delta\boldsymbol{\gamma})} \quad (16)$$

where $\bar{F}_i'(\Delta\boldsymbol{\gamma})$ is the derivative of the residual function with respect to $\Delta\gamma_i$.

2.3 Numerical implementation

We can now define the final algorithm for the nonlocal constitutive state update. However, since we are using an elastic predictor/return mapping scheme, equation (11) is only conceptually valid for plastic points. Otherwise, the material point is elastic and no return mapping is necessary. In order to overcome this shortcoming, Strömberg and Ristinmaa [7] have proposed a strategy that checks the yield condition for each material point on every Newton-Raphson iteration and, when $\bar{F}_i \leq 0$ and $\Delta\gamma_i^{k+1} = 0$, $\bar{F}_i$ is set to 0. It is important to remark that it is not (usually) possible to know *a priori* which points are elastic or plastic. Therefore, a point which is initially elastic, can turn into plastic and then elastic again as iterations evolve until final convergence is reached. The final algorithm for the nonlocal constitutive integration is summarised in boxes 1 and 2.

***Box 1**. Nonlocal stress updating procedure for the simplified Lemaitre's model under small strains.*

(i) Define elastic trial state for every Gauss point as

$$\boldsymbol{\varepsilon}_{n+1}^{e\,trial} = \boldsymbol{\varepsilon}_n^e + \Delta\boldsymbol{\varepsilon}; \qquad \tilde{\boldsymbol{s}}_{n+1}^{trial} = 2G\boldsymbol{\varepsilon}_{d\,n+1}^{e\,trial}$$

$$\bar{R}_{n+1}^{trial} = \bar{R}_n; \qquad q_{n+1}^{trial} = \sqrt{\frac{3}{2}}\frac{\left\|\tilde{\boldsymbol{s}}_{n+1}^{trial}\right\|}{(1-D_n)}$$

$$\tilde{p}_{n+1} = K\varepsilon_{v\,n+1}^{e\,trial};$$

(ii) `GOTO BOX 2` to solve

$$\bar{\boldsymbol{F}}(\Delta\boldsymbol{\gamma}) = \boldsymbol{0}$$

for $\Delta\boldsymbol{\gamma}$ with the Newton-Raphson method.

(iii) Update stress state for every Gauss point as

$$D_{n+1} = 1 - \left(\frac{3G\Delta\gamma}{q_{n+1}^{trial} - \sigma_y(\bar{R}_{n+1})}\right)$$

$$p_{n+1} = (1 - D_{n+1})\tilde{p}_{n+1}$$

$$q_{n+1} = (1 - D_{n+1})\sigma_y(\bar{R}_{n+1})$$

$$\boldsymbol{s}_{n+1} = \frac{q_{n+1}}{q_{n+1}^{trial}}\tilde{\boldsymbol{s}}_{n+1}^{trial}$$

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{s}_{n+1} + p_{n+1}\boldsymbol{I}$$

$$\boldsymbol{\varepsilon}_{n+1}^e = \frac{1}{2G}\boldsymbol{s}_{n+1} + \frac{1}{3K}\varepsilon_{v\,n+1}^{e\,trial}\boldsymbol{I}$$

(iv) `EXIT`

Box 2. *Newton-Raphson solution for the nonlocal simplified Lemaitre's damage model.*

(i) Initialise counter $k = 0$ and $\Delta\boldsymbol{\gamma}^k = \Delta\boldsymbol{\gamma}^{(0)} = \mathbf{0}$

(ii) Evaluate initial residual and its derivative for all points

FOR I = 1, NGAUSP

IF $\tilde{q}_i^{trial} - \sigma_{y_i}(\bar{R}_i^k) \leq 0$ THEN

$\bar{F}_i = 0$

ELSE

$$\bar{F}_i(\Delta\gamma_i^k) = 3G_i\Delta\gamma_i^k - (1 - D_{i_n})[\tilde{q}_i^{trial} - \sigma_{y_i}(\bar{R}_i^k)] + \frac{[\tilde{q}_i^{trial} - \sigma_{y_i}(\bar{R}_i^k)]^2}{3G_i}\left(\frac{-Y_i}{r}\right)^s$$

$$\bar{F}_i'(\Delta\gamma_i^k) = 3G_i + w_iJ_i\alpha_{ii}H|_{\bar{R}_i^k} - \frac{2}{3G_i}[\tilde{q}_i^{trial} - \sigma_{y_i}(\bar{R}_i^k)]w_iJ_i\alpha_{ii}H|_{\bar{R}_i^k}\left(\frac{-Y_i}{r}\right)^s$$

$$+\frac{s}{r}\left(\frac{-Y_i}{r}\right)^{s-1}\frac{\sigma_{y_i}(R_i^k)w_iJ_i\alpha_{ii}H|_{\bar{R}_i^k}}{9G_i^2}[\tilde{q}_i^{trial} - \sigma_{y_i}(\bar{R}_i^k)]$$

ENDIF

END LOOP

(iii) Update incremental plastic multiplier for every point as

$$\Delta\gamma_i^{k+1} = \Delta\gamma_i^k - \frac{\bar{F}_i(\Delta\gamma_i^k)}{\bar{F}_i'(\Delta\gamma_i^k)}$$

(iv) Update the nonlocal isotropic hardening variable for every Gauss point as

$$\bar{R}_i^{k+1} = \bar{R}_i^{(0)} + \sum_{j=1}^{n_{gpi}} w_jJ_j\alpha_{ij}\Delta\gamma_j^{k+1}$$

(v) Evaluate residual and its derivative for all points

FOR I = 1, NGAUSP

IF $\tilde{q}_i^{trial} - \sigma_{y_i}(\bar{R}_i^{k+1}) \leq 0$ AND $\Delta\gamma_i^{k+1} = 0$ THEN

$\bar{F}_i = 0$

ELSE

$$\bar{F}_i(\Delta\gamma_i^{k+1}) = 3G_i\Delta\gamma_i^{k+1} - (1 - D_{i_n})[\tilde{q}_i^{trial} - \sigma_{y_i}(\bar{R}_i^{k+1})] + \frac{[\tilde{q}_i^{trial} - \sigma_{y_i}(\bar{R}_i^{k+1})]^2}{3G_i}\left(\frac{-Y_i}{r}\right)^s$$

$$\bar{F}_i'(\Delta\gamma_i^{k+1}) = 3G_i + w_iJ_i\alpha_{ii}H|_{\bar{R}_i^{k+1}} - \frac{2}{3G_i}[\tilde{q}_i^{trial} - \sigma_{y_i}(\bar{R}_i^{k+1})]w_iJ_i\alpha_{ii}H|_{\bar{R}_i^{k+1}}\left(\frac{-Y_i}{r}\right)^s$$

$$+\frac{s}{r}\left(\frac{-Y_i}{r}\right)^{s-1}\frac{\sigma_{y_i}(\bar{R}_i^{k+1})w_iJ_i\alpha_{ii}H|_{\bar{R}_i^{k+1}}}{9G_i^2}[\tilde{q}_i^{trial} - \sigma_{y_i}(\bar{R}_i^{k+1})]$$

ENDIF

END LOOP

(vi) Check convergence

IF $\|\bar{\boldsymbol{F}}\| < TOL$ EXIT

ELSE GOTO (iii)

3. NUMERICAL ANALYSIS

In this section a numerical example of a body in plane strain subject to a compressive load (see figure 1) is presented to assess the efficiency of the adopted numerical scheme. The Young's modulus, E, of the material employed in the simulation is equal to $20\ MPa$ and the Poisson's ratio, ν, is equal to 0.49. The yield stress, σ_y, is equal to 20 MPa and the hardening modulus, H, is equal to 2.0 MPa. A small area at the bottom left has a 10% lower yield stress in order to trigger localisation. Quadratic elements with 8 nodes, with reduced integration, were used in the analysis for three different meshes depicted in figure 2.

A high value of the material parameter r(denominator) was used in order to reproduce as a limiting case the von Mises plasticity theory enhanced with the nonlocal strategy. Figures 2, 3 and 4 show the results for both local and nonlocal analysis. The results show that the dependency of results upon the mesh refinement was attenuated. A typical convergence of the stress update procedure with the nonlocal strategy is depicted in table 1. As reported by Strömberg & Ristinmaa [7], convergence was attained within 3 iterations.

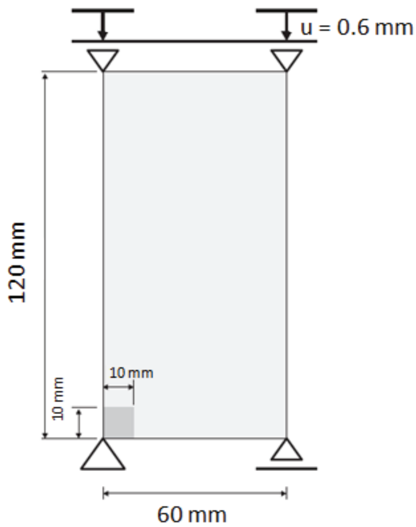

Fig. 1. *Plane strain analysis of a plate under compressive loading.*

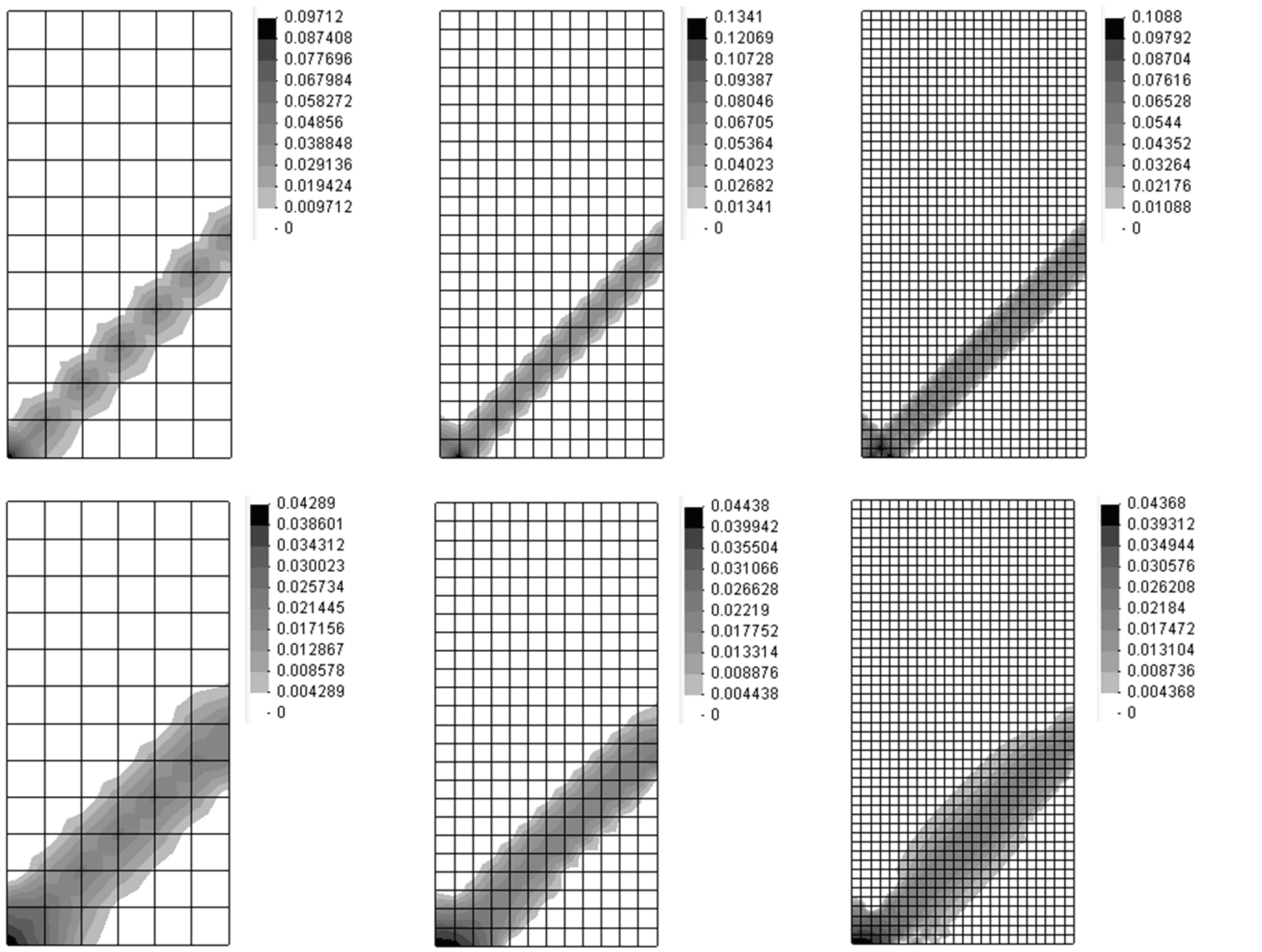

Fig. 2. *Isotropic hardening variable contours for the local (above) and nonlocal (below) formulations.*

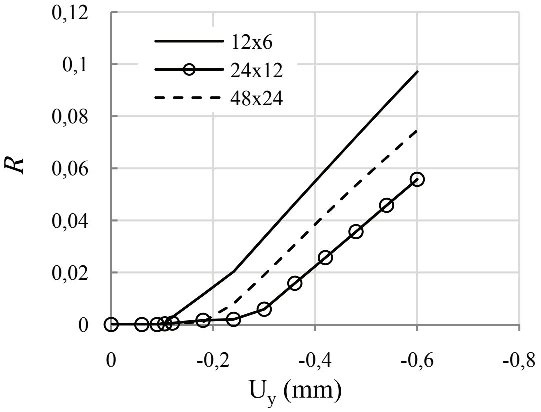

Fig. 3. *Isotropic hardening variable evolution on the lower left corner for the local solution.*

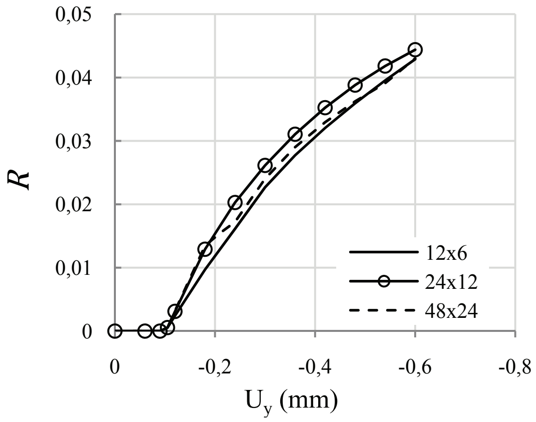

Fig. 4. *Isotropic hardening variable evolution on the lower left corner for the nonlocal solution.*

Table 1. *Typical convergence of the algorithm.*

Iteration	Residual norm
1	2.09 E + 000
2	9.88 E − 005
3	5.64 E − 009

4. CONCLUSIONS

A finite element framework for a nonlocal damage model was presented. The regularisation of the solution was achieved by replacing the local isotropic hardening variable with a nonlocal definition employing a nonlocal integral strategy. The results show that the presented methodology acts as localisation limiter for the examples tested. Furthermore, the numerical implementation of the scheme in existing FE codes is relatively straightforward since only the internal force computation is affected.

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NIELOKALNE CAŁKOWE SFORMUŁOWANIE MODELU PĘKANIA INDUKOWANEGO ODKSZTAŁCENIEM PLASTYCZNYM

Streszczenie

Predykcja uszkodzenia indukowanego odkształceniem plastycznym powoduje zwykle konieczność wykorzystania metody elementów skończonych w zakresie sprężysto-plastycznym z modelem uszkodzeń. W miejscu gdzie tworzy się lokalizacja odkształcenia, cząstkowe równania różniczkowe tracą eliptyczności, co powoduje że rozwiązania numeryczne stają się zależne od siatki elementów. Jednym z rozwiązań tego problemu jest wprowadzenie wewnętrznej długości, która odpowiada zadanej wielkości mikrostrukturalnej materiału. W takim rozwiązaniu nielokalne sformułowania całkowe oraz rozszerzone sformułowania gradientowe są powszechnie wykorzystywane jako strategie regularyzacji. W niniejszej pracy sformułowanie metody elementów skończonych oparte o nielokalny schemat całkowania zaproponowany przez Strömberga & Ristinmaa (1996) rozszerzone zostało o uproszczoną wersję modelu uszkodzeń indukowanych plastycznie Lemaitre'a. Jako, że model ten ma wpływ jedynie na obliczenie sił wewnętrznych, jego implementacja w istniejących kodach MES jest stosunkowo prosta. Otrzymane wyniki z przykładowych obliczeń pokazują, iż zaimplementowana strategia pozwala na pokonanie problemów związanych z lokalizacją odkształceń.

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