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# INTERDIFFUSION IN R-COMPONENT (R ≥ 2) ONE DIMENSIONAL MIXTURE SHOWING CONSTANT CONCENTRATION

MAREK DANIELEWSKI[1], KONSTANTY HOLLY[2], WOJCIECH KRZYŻAŃSKI[3]

[1]University of Mining and Metallurgy, Faculty of Materials Science and Ceramics, ul. Mickiewicza 30, 30-059 Cracow, Poland
[2]Jagiellonian University, Institute of Mathematics, ul. Reymonta 4, 30-059 Cracow, Poland
[3]SUNY at Buffalo, 547 Cooke Hall, Buffalo, NY 14260-1200, USA
Corresponding Author: daniel@agh.edu.pl (M. Danielewski)

**Abstract**

It is shown that multicomponent nonequilibrium system relaxing by a diffusion process exhibit unique behavior during evolution toward stationary state. The Darken's postulate that the total mass flow is a sum of diffusion flux and translation are applied for the general case of diffusional transport in r-component solution (process defined as interdiffusion in unidimensional mixture). The equations of local mass conservation (continuity equations), the Darken's flux formulas, the postulate of constant molar volume of the mixture (valid e.g. in solid solutions) and the initial and boundary conditions form a self-consistent interdiffusion problem. The variational form of the interdiffusion problem is derived. It has been proved that: i) there exists a weak solution of variational formulation of interdiffusion problem in open as well as in closed systems, ii) a weak solution of variational formulation of interdiffusion problem is the unique solution, iii) the standard deviation of molar ratio of mixture components in the closed system is monotonically decreasing to 0 when time $t \to \infty$ (i.e. mixture becomes homogeneous) and iv) in closed system the gradients of density of every mixture component vanish at the mixture boundaries.

**Key words**: material

## 1. INTRODUCTION

The theoretical framework for the description of nonequilibrium systems is essential for a wide range of disciplines in both science and technology. Thus it is useful to study models describing the complexity of the real system. The key models of the interdiffusion base on the Onsager or Darken approach. The Onsager method was a generalization of the Nernst-Planck formulae (Nernst, 1888; Nernst, 1889; Planck, 1890a; Planck, 1890b). It was widely applied for ternary and quaternary closed systems. The universality of the Onsanger model has become the subject of long-standing debate (Danielewski et al., 2008). Its discussion is beyond the aim of this work. Instead we will show that Darken method allows to overcome the limitations of previous models.

The majority of phenomenological models of the interdiffusion have neglected the effects due to variation of medium properties with the composition. The fundamental Darken and Wagner equations are limited to the two component systems and assume that the partial molar volumes of the diffusing components are constant (Wagner, 1951; Darken, 1948) and equal (Darken, 1948). Some aspects of the Fick or Nernst-Planck flux formulae and continuity equation, that are a core of the Darken model, require closer examination.

Fick's and more general Nernst-Planck constitutive flux formulae (Nernst, 1888; Nernst, 1889;


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Planck, 1890a; Planck, 1890b) are equivalent to each other in a case of ideal solution. The Nernst-Planck formula alows to extend Darken model in a case of non-ideal alloys, oxides and ionic solutions. These equations and their application can be criticized for several reasons (Buck, 1984). The main reasons are the "macroscopic, smooth nature of the model used in its derivation" and the omission of "cross terms". Despite these objections, they show usefulness and wide aplicability. These objections seem to be of little importance in the case of diffusional couple, since such couples are relatively thick (with respect to Debye length) and homogenous. *Continuity equation* is undoubtedly one of the main and unchallenged laws of physics. It cannot be reasonably disputed. It provides the basic relation between space and time changes of concetration of all elements. Here it is used in reduced form. Namely, the reaction terms (sink and/or source of mass) are neglected, i.e. we do not consider the new phase and/or species formation within diffusion zone.

This paper is an attempt to extend the Darken interdiffusion phenomenology. The essence of this attempt incorporate the postulate that the total mass flow (the Darken's flux) is a sum of diffusion and translation fluxes.

A more detailed analysis of the concepts of drift velocity, the choice of the proper reference frame for diffusion, as well as the other consequences of the proposed formalism have already been published (Danielewski et al., 1994b; Danielewski et al., 1994a; Holly et al., 1994).

A general phenomenological treatment of the interdiffusion problem is given below. No assumptions whatsoever are made as far as the mechanism of the diffusion processes is concerned.

In this work it will be proved that: i) there exists a weak solution of variational formulation of interdiffusion problem in open as well as in closed system, ii) a weak solution of variational formulation of interdffusion problem is the unique solution, iii) the standard deviation of molar ratio of mixture components in the closed system is monotonically decreasing to 0 when time $t \to \infty$ (i.e., the mixture becomes homogeneous) and iv) in closed system the gradients of density of every mixture component vanish at the mixture boundaries.

## 2. MODEL OF INTERDIFFUSION

We consider a mixture of $r$-components ($r \geq 2$) having the densities $\varrho_1, \ldots, \varrho_r$ and molar masses $M_1, \ldots, M_r$. Let us assume that components are placed in the space delimitated by the surfaces at the positions $-\Lambda,\ \Lambda$ where $\Lambda > 0$.

As a result of diffusion the densities $\varrho_1, \ldots, \varrho_r$ are altered all over the time. Thus we can write for $i = 1, \ldots, r$:

$$\varrho_i : [0,\, T) \times [-\Lambda,\, \Lambda] \to [0,\, \infty).$$

By $\overset{\circ}{\varrho}_i$ we denote the density of the $i$-th component at the moment $t = 0$ (the initial $i$-th component distribution) for $x \in [-\Lambda,\, \Lambda],\ i = 1, \ldots, r$:

$$\overset{\circ}{\varrho}_i(x) = \varrho_i(0,\, x). \tag{1}$$

The diffusion process generates movement of each component which is characterised by the velocity field $v_i$:

$$v_i : [0,\, T) \times [-\Lambda,\, \Lambda] \to \mathbb{R}.$$

where $\mathbb{R}$ denotes a set of real numbers. In general, a flow of mass can occur through the left $(-\Lambda)$ and the right boundary $(\Lambda)$ for $t \in [0,\, T),\ i = 1, \ldots, r$:

$$j_{i,\,L}(t) = (\varrho_i v_i)(t,\, \Lambda),\ j_{i,\,R}(t) = (\varrho_i v_i)(t,\, \Lambda). \tag{2}$$

Thus for $i = 1, \ldots, r$

$$j_{i,\,L}, j_{i,\,R} : [0,\, T) \to \mathbb{R}.$$

When the interdiffusion process occurs in a closed system, the mass can not be transported through the boundary, then for $t \in [0,\, T),\ i = 1, \ldots, r$

$$j_{i,\,L}(t) = j_{i,\,R}(t) = 0. \tag{3}$$

The total mass of the $i$-th component of the mixture at the fixed moment $t$ is given by:

$$m_i(t) = \int_{-\Lambda}^{\Lambda} \varrho_i(t,\, x)\, dx. \tag{4}$$

The total mass of the mixture at the moment $t$ is the sum:

$$m_1(t) + \ldots + m_r(t) = \\ = \int_{-\Lambda}^{\Lambda} \big(\varrho_1(t,\, x) + \ldots + \varrho_r(t,\, x)\big)\, dx.$$

Accordingly, the local average density in the mixture is the sum of the densities of the components

$$\varrho = \varrho_1 + \ldots + \varrho_r.$$

**Physical laws.**

**1.** We assume that each component of the mixture is a continuous medium, i.e. it satisfies the local mass conservation equation:

$$\frac{\partial \varrho_i}{\partial t} + \frac{\partial}{\partial x}(\varrho_i v_i) = 0 \quad (i = 1, \ldots, r). \tag{5}$$

**2.** Following the Darken's concept of the drift velocity in solids (Darken, 1948) we postulate that for $i = 1, \ldots, r$

$$\varrho_i v_i = -\Theta_i(\varrho_1, \ldots, \varrho_r)\frac{\partial \varrho_i}{\partial x} + \varrho_i v \tag{6}$$

$$\begin{bmatrix}\text{the Darken's} \\ \text{flow}\end{bmatrix} = \begin{bmatrix}\text{the diffusional} \\ \text{flux}\end{bmatrix} + \begin{bmatrix}\text{the Darken} \\ \text{drift flux}\end{bmatrix}$$

where $v$ denotes the local drift velocity of the mixture

$$v : [0, T) \times [-\Lambda, \Lambda] \to \mathbb{R}$$

and $\Theta_i$ is a given diffusion coefficient of the $i$-th component in the mixture. In general, $\Theta_i$ depends on $\varrho_1$, ..., $\varrho_r$

$$\Theta_i : \underbrace{[0, \infty) \times \ldots \times [0, \infty)}_{r\ times} \to (0, \infty)$$

for $i = 1, \ldots, r$. Further we assume that $\Theta_i$ satisfies the Lipschitz condition.

**3.** The postulate of constant concentration of the mixture:

$$\frac{\varrho_1}{M_1} + \ldots + \frac{\varrho_r}{M_r} = c, \tag{7}$$

where $c \in (0, \infty)$ is the concentration of the mixture.

From the law of mass conservation (**1.**) and conditions (1)-(3) and (4) we get for $t \in [0, T)$ and $i = 1, \ldots, r$:

$$m_i(t) = \int_{-\Lambda}^{\Lambda} \overset{\circ}{\varrho}_i(x)\,dx + \int_0^t \big(j_{i,L} - j_{i,R}\big)(\tau)\,d\tau. \tag{8}$$

In the closed system the total mass of each component does not depend on time, consequently for $t \in [0, T)$ and $i = 1, \ldots, r$

$$m_i(t) = \int_{-\Lambda}^{\Lambda} \overset{\circ}{\varrho}_i(x)\,dx = \mathbf{const}. \tag{9}$$

The consequence of the law of mass conservation (**1.**) and the postulate of constant concentration (**3.**) is that for any $t \in [0, T)$ and $x \in [-\Lambda, \Lambda]$

$$\frac{\partial}{\partial x}\left(\sum_{i=1}^{r} \frac{(\varrho_i v_i)(t, x)}{cM_i}\right) = 0. \tag{10}$$

Consequently, from (2) and (9) it follows that at the boundaries

$$\sum_{i=1}^{r} \frac{\big(j_{i,L}(t) - j_{i,R}(t)\big)}{M_i} = 0. \tag{11}$$

Equation (7) implies that

$$\overset{\circ}{\varrho}_i(x) \in [0, cM_i] \quad (i = 1, \ldots, r). \tag{12}$$

The initial, boundary condition, physical laws and derived equations represent the complete description of the unidimensional $r$-component mixture of constant volume and position. The key differences between the Darken's model, the numerous works of his followers and present approach are: **1)** the mixture composition ($r > 2$), **2)** the finite dimensions and finally **3)** the mass transport through mixture boundaries.

## 3. FORMULATION OF THE BOUNDARY VALUE PROBLEM OF INTERDIFFUSION IN THE $r$-COMPONENT ONE DIMENSIONAL SOLID SOLUTION

In this section the problem of interdiffusion in a $r$-component solid solution will be formulated.

**Data:**

1. $M_i \in (0, \infty)$ - molecular mass of the $i$-th component of the mixture, e.g. an alloy, $(i = 1, \ldots, r)$;
2. $\Lambda \in (0, \infty)$ - right boundary of the segment occupied by the mixture;
3. $\overset{\circ}{\varrho}_i : [-\Lambda, \Lambda] \to [0, \infty)$ - initial density distribution of the $i$-th component in the mixture $(i = 1, \ldots, r)$. We assume that the initial global concentration of the mixture is constant and positive, i.e. $\overset{\circ}{\varrho}_i$ satisfies (7);
4. $\Theta_i : [0, cM_1] \times \cdots \times [0, cM_r] \to (0, \infty)$ - diffusion coefficient of $i$-th component $(i = 1, \ldots, r)$, that satisfies the Lipschitz condition;

5. $T \in (0, \infty)$ - the examination time (time at which measurements were carried out);

6. $j_{i,L} : [0, T] \to \mathbb{R}$ - mass flow of the $i$-th component through the left boundary ($i = 1, \ldots, r$);

7. $j_{j,R} : [0, T] \to \mathbb{R}$ - mass flow of the $j$-th component through the right boundary ($j = 1, \ldots, r$).

We assume that for any $t \in [0, T]$ the total mass of each component is nonnegative:

$$m_i(t) = \int_{-\Lambda}^{\Lambda} \mathring{\varrho}_i(x)\, dx + \int_0^t \left(j_{i,L} - j_{i,R}\right) d\tau \geq 0, \tag{13}$$

for $i = 1, \ldots, r$.

Also, the mass flows at the boundaries are constrained by (11).

The solutions of the boundary value problem of interdiffusion in the $r$-component solid solution are the functions $\varrho_i$ ($i = 1, \ldots, r$) and $v$ such that the physical laws (5), (6), (7), the initial and boundary conditions (1) and (2) are satisfied.

## 4. VARIATIONAL FORM OF THE PROBLEM

Upon multiplying by $\frac{1}{cM_i}$ the total flux of the $i$-th component (6) and adding all the obtained equations, one gets:

$$\frac{\partial}{\partial x}\left(v - \sum_{i=1}^{r} \frac{\Theta_i(\varrho_1, \ldots, \varrho_r)}{cM_i} \frac{\partial \varrho_i}{\partial x}\right) = 0. \tag{14}$$

Consequently for any time $t \in [0, T]$ there exists a unique $K(t) \in \mathbb{R}$ such that for any $x \in [-\Lambda, \Lambda]$:

$$v(t, x) = K(t) + \sum_{i=1}^{r}\left(\frac{\Theta_i(\varrho_1, \ldots, \varrho_r)}{cM_i} \frac{\partial \varrho_i}{\partial x}\right)(t, x). \tag{15}$$

where $K(t)$ represents the barycentric combination of all mass fluxes in the mixture:

$$K(t) = \sum_{i=1}^{r} \frac{\varrho_i}{cM_i} v_i. \tag{16}$$

Thus, $K(t)$ can be understood as the field of local velocities of mixture forming particles and can be calculated from the boundary condition (2):

$$K(t) = \sum_{i=1}^{r} \frac{j_{i,L}(t)}{cM_i} \left(= \sum_{i=1}^{r} \frac{j_{i,R}(t)}{cM_i}\right). \tag{17}$$

Thus, (15) reduces the set of unknown functions of the problem to $\varrho_1, \ldots, \varrho_r$.

For $t \in [0, T]$ and $x \in [-\Lambda, \Lambda]$ we define the deviation of the molar ratio, $w_i(t, x)$, from its average value in the mixture, $\mathfrak{m}_i(t)$:

$$w_i(t, x) = \frac{\varrho_i(t, x)}{cM_i} - \mathfrak{m}_i(t), \tag{18}$$

where

$$\mathfrak{m}_i(t) := \frac{m_i(t)}{2\Lambda c M_i}. \tag{19}$$

The following identities follow from (4) for any $t \in [0, T]$

$$\sum_{i=1}^{r} \mathfrak{m}_i(t) = 1, \tag{20}$$

$$\sum_{i=1}^{r} w_i \equiv 0, \tag{21}$$

$$\int_{-\Lambda}^{\Lambda} w_i(t, x)\, dx = 0. \tag{22}$$

Equations (20) and (21) justify introducing the following numeric space

$$\mathbb{I}^{\perp} = \{w \in \mathbb{R}^r \mid w_1 + \ldots + w_r = 0\}.$$

Using (18) and (19) one can express $\varrho_i(t)$ in terms of $w_i(t)$ and $m_i(t)$:

$$\varrho_i(t, x) = c\, M_i\, w_i(t, x) + \frac{m_i(t)}{2\,\Lambda}. \tag{23}$$

For such rescaled densities the diffusion coefficient $\Theta_i(\varrho_1, \ldots, \varrho_r)$ takes on the following form

$$\Theta_i\left(\varrho_1 \ldots, \varrho_r\right) = \tilde{\Theta}_i\left(w_1 + \mathfrak{m}_1, \ldots, w_r + \mathfrak{m}_r\right) \tag{24}$$

where

$$\mathfrak{m}(t) := \Big(\mathfrak{m}_1(t), \ldots, \mathfrak{m}_r(t)\Big).$$

For the sake of further analysis we introduce functions of time that attain values in the space of functions of $x$ variable defined as $w(t)(x) = w(t, x)$, etc. Equation (6) can be now written as follows

$$\frac{\left(\varrho_i v_i\right)(t)}{cM_i} = -\left(\tilde{\Theta}_i \circ \Big(w(t) + \mathfrak{m}(t)\Big)\right)\frac{\partial w_i(t)}{\partial x} + K(t)\Big(w_i(t) + \mathfrak{m}_i(t)\Big) + \Big(w_i(t) + \mathfrak{m}_i(t)\Big)\sum_{k=1}^{r}\left(\tilde{\Theta}_k \circ \left(w(t) + \mathfrak{m}(t)\right)\right)\frac{\partial w_k(t)}{\partial x} \tag{25}$$

where the symbol "∘" denotes the superposition of two functions.

For $\kappa \in \mathbb{R}^r$ such that $\kappa_1, \ldots, \kappa_r \geq 0$ and $\kappa_1 + \ldots + \kappa_r = 1$ one can define the operator $A_\kappa : \mathbb{I}^\perp \to \mathbb{R}^r$

$$(A_\kappa \xi)_i = \tilde{\Theta}_i(\kappa)\xi_i - \kappa_i \sum_{k=1}^{r} \tilde{\Theta}_k(\kappa)\xi_k \,, \tag{26}$$

for $\xi \in \mathbb{I}^\perp$ and $i = 1, \ldots, r$. Consequently, equation (25) can be abbreviated to

$$\frac{(\varrho_i v_i)(t)}{cM_i} = -\left(A_{w(t)+\mathfrak{m}(t)} \frac{\partial w(t)}{\partial x}\right)_i + K(t)\Big(w_i(t) + \mathfrak{m}_i(t)\Big). \tag{27}$$

The equations (5) and definition (18) imply that

$$\frac{\partial w_i}{\partial t}\Big(t, x\Big) = -\frac{\partial}{\partial x}\frac{(\varrho_i v_i)(t,\, x)}{cM_i} - \overset{\bullet}{\mathfrak{m}}_i(t); \tag{28}$$

Where $\overset{\bullet}{\mathfrak{m}}_i$ denotes the time derivative of $\mathfrak{m}_i$. Analogously,

$$\frac{\partial w_i(t,\, x)}{\partial x} = \frac{1}{cM_i}\frac{\partial \varrho_i(t,\, x)}{\partial x}. \tag{29}$$

Upon introducing $\phi : [-\Lambda,\, \Lambda] \to \mathbb{I}^\perp$, $\int_{-\Lambda}^{\Lambda} \phi\, dx = 0$. We can multiply (28) by $\phi_i$, and integrate over the range $-\Lambda \leq x \leq \Lambda$ with the result:

$$\int_{-\Lambda}^{\Lambda} \frac{\partial w_i}{\partial t}(t,\, x)\phi_i(x)\, dx = \int_{-\Lambda}^{\Lambda}\left(\frac{\partial}{\partial x}\frac{(\varrho_i v_i)(t)}{cM_i}\right)\phi_i\, dx - \int_{-\Lambda}^{\Lambda} \overset{\bullet}{\mathfrak{m}}_i(t)\phi_i\, dx. \tag{30}$$

Since $\mathfrak{m}_i(t)$ does not depend on $x$, the last term in (30) equals zero and it follows that:

$$\begin{aligned}\frac{d}{dt}\int_{-\Lambda}^{\Lambda} w_i(t)\phi_i\, dx = &-\int_{-\Lambda}^{\Lambda}\left(A_{w(t)+\mathfrak{m}(t)}\frac{\partial w(t)}{\partial x}\right)_i \frac{\partial \phi_i}{\partial x}\, dx + \\ &K(t)\int_{-\Lambda}^{\Lambda} w_i(t)\frac{\partial \phi_i}{\partial x}\, dx + \\ &\phi_i(\Lambda)\left(K(t)\,\mathfrak{m}_i(t) - \frac{j_{i,\,R}(t)}{cM_i}\right) - \\ &\phi_i(-\Lambda)\left(K(t)\,\mathfrak{m}_i(t) - \frac{j_{i,\,L}(t)}{cM_i}\right).\end{aligned} \tag{31}$$

Upon denoting

$$\Gamma_R(t)_i = K(t)\,\mathfrak{m}_i(t) - \frac{j_{i,\,R}(t)}{cM_i},$$

$$\Gamma_L(t)_i = K(t)\,\mathfrak{m}_i(t) - \frac{j_{i,\,L}(t)}{cM_i}$$

and noting that:

$$\sum_{i=1}^{r} \Gamma_R(t)_i = \sum_{i=1}^{r} \Gamma_L(t)_i = 0.$$

For all mixture components, $i = 0, \ldots, r$, equations (31) can be written in the vector form:

$$\begin{aligned}\frac{d}{dt}\int_{-\Lambda}^{\Lambda} w(t)\phi\, dx = &-\int_{-\Lambda}^{\Lambda}\left(A_{w(t)+\mathfrak{m}(t)}\frac{\partial w(t)}{\partial x}\right)\frac{\partial \phi}{\partial x}\, dx \\ &+K(t)\int_{-\Lambda}^{\Lambda} w(t)\frac{\partial \phi}{\partial x}\, dx + \phi(\Lambda)\Gamma_R(t) - \phi(-\Lambda)\Gamma_L(t),\end{aligned} \tag{32}$$

where

$$\Gamma_L(t) = (\Gamma_L(t)_1, \ldots, \Gamma_L(t)_r)\left(\in \mathbb{I}^\perp\right), \tag{33}$$

$$\Gamma_R(t) = (\Gamma_R(t)_1, \ldots, \Gamma_R(t)_r)\left(\in \mathbb{I}^\perp\right). \tag{34}$$

Let $\left(\mathbb{I}^\perp\right)^{\{-\Lambda,\,\Lambda\}}$ denote the space of all functions $\{-\Lambda,\, \Lambda\} \to \mathbb{I}^\perp$, then for any $t \geq 0$

$$\Gamma(t) := \{(-\Lambda,\, \Gamma_L(t))\,,\, (\Lambda,\, \Gamma_R(t))\} \in \left(\mathbb{I}^\perp\right)^{\{-\Lambda,\,\Lambda\}}, \tag{35}$$

and

$$\phi \cdot \Gamma(t)\Big|_{-\Lambda}^{\Lambda} = \phi(\Lambda)\Gamma_R(t) - \phi(-\Lambda)\Gamma_L(t).$$

Consequently, integration of (32) in the time range $\Big|_0^t$ results in:

$$\begin{aligned}\Big(w(t)\Big|\phi\Big)_{L^2} - \Big(\overset{\circ}{w}\Big|\phi\Big)_{L^2} = &-\int_0^t\int_{-\Lambda}^{\Lambda}\left(A_{w(\tau)+\mathfrak{m}(\tau)}\frac{\partial w(\tau)}{\partial x}\right)\frac{\partial \phi}{\partial x}\, dx\, d\tau + \\ &\int_0^t K(\tau)\left(w(\tau)\Big|\frac{\partial \phi}{\partial x}\right)_{L^2} d\tau + \int_0^t \phi \cdot \Gamma(\tau)\Big|_{-\Lambda}^{\Lambda} d\tau.\end{aligned} \tag{36}$$

where $(\cdot|\cdot)_{L^2}$ denotes the inner product in the real Hilbert space $L^2(-\Lambda,\, \Lambda;\, \mathbb{R}^r)$ (Adams, 1975).

We assume the following regularity of fluxes for $i = 1, \ldots, r$

$$j_{i,\,L} \in L^2(0,\, T) \quad \text{and} \quad j_{i,\,R} \in L^2(0,\, T). \tag{37}$$

From (7) it follows that

$$\kappa \in L^2(0,\, T) \tag{38}$$

and (13) and (19) imply that

$$\mathfrak{m} \in H^1(0,\, T;\, \mathbb{R}^r) \tag{39}$$

where $H^1(0,\, T;\, \mathbb{R}^r)$ denotes the real Sobolev space (Adams, 1975).

Consequently,

$$\Gamma_L \in L^2\left(0,\, T;\, \mathbb{I}^{\perp}\right) \text{ and } \Gamma_R \in L^2\left(0,\, T;\, \mathbb{I}^{\perp}\right). \tag{40}$$

Our next step is to specify regularity conditions for $w$ and $\phi$ such that the integrals in (36) are well defined. To do so we introduce the following spaces

$$H = \left\{ \psi \in L^2(-\Lambda,\, \Lambda;\, \mathbb{I}^{\perp}) \mid \int_{-\Lambda}^{\Lambda} \psi_i \, dx = 0 \quad \text{for} \quad i = 1, \ldots, r \right\}. \tag{41}$$

and

$$V = \left\{ \psi \in H^1(-\Lambda,\, \Lambda;\, \mathbb{R}^r) \mid \psi \in H \right\}. \tag{42}$$

The space $V$ with the scalar product

$$((\phi,\, \psi)) := \sum_{i=1}^{r} \int_{-\Lambda}^{\Lambda} \frac{\partial \phi_i}{\partial x} \frac{\partial \psi_i}{\partial x} \, dx \tag{43}$$

is a Hilbert space and the inclusion $V \hookrightarrow H$ is continuous (Adams, 1975).

Equation (36) is a weak formulation of the problem (5)-(7) with the boundary and initial conditions (1)-(3). Below we present general assumptions about the regularity of $w$ such that (36) is well defined.

**Definition 4.1.** A function $w : [0,\, T] \to H$ is a weak solution of the boundary problem of interdiffusion in the $r$-component one dimensional solid solution if

$$w \text{ is weakly continuous,} \tag{44}$$

$$\text{for almost all } t \in [0,\, T] \;:\; w(t) \in V, \tag{45}$$

$$w \in L^2(0,\, T;\, V), \tag{46}$$

$$\text{for any } t \in [0,\, T] \text{ and any } \phi \in V \text{ (36) holds.} \tag{47}$$

The condition (45) means that the set of $t \in [0,\, T]$ for which $w(t) \notin V$ has the zero Lebesgue measure (Rudin, 1974). The space $L^2(0,\, T;\, V)$ in (46) is defined in (Dautray et al., 1988).

The existence of the integral $\int_0^t \phi \cdot \Gamma(\tau)\big|_{-\Lambda}^{\Lambda} \, d\tau$ for any $t \in [0,\, \tau]$ is a consequence of (40). (45) implies that the function $\bar{w}(t) : [0,\, T] \to V$:

$$\bar{w}(t) = \begin{cases} w(t); & \text{if } w(t) \in V \\ 0; & \text{if } w(t) \notin V \end{cases}$$

is Borel measurable. Equation (39) yields that the function

$$\bar{w} + \mathfrak{m} : [0,\, T] \to H^1(-\Lambda,\, \Lambda;\, \mathbb{R}^r)$$

is Borel measurable. From **Lemma A.1** proved in **Appendix A** it follows that

$$\left\| A_{\bar{w}(\tau)+\mathfrak{m}(\tau)} \frac{\partial \bar{w}(\tau)}{\partial x} \right\|_{L^2} \leq \|A\|_{L^\infty} \cdot \left\| \frac{\partial \bar{w}(\tau)}{\partial x} \right\|_{L^2}. \tag{48}$$

Since

$$\left\| \frac{\partial \bar{w}(\tau)}{\partial x} \right\|_{L^2} \leq \|\bar{w}(\tau)\|_V$$

The condition (46) implies that for any $t \in [0,\, T]$ and $\phi \in V$

$$\int_0^t \int_{-\Lambda}^{\Lambda} \left| \left( A_{\bar{w}(\tau)+\mathfrak{m}(\tau)} \frac{\partial \bar{w}(\tau)}{\partial x} \right) \frac{\partial \phi}{\partial x} \right| dx \, d\tau < \infty$$

and consequently the first integral in (36) is well defined.

Similarly, (38) and (46) imply that for any $t \in [0,\, T]$ and $\phi \in V$

$$\int_0^t \left| \kappa(\tau) \left( w(\tau) \Big| \frac{\partial \phi}{\partial x} \right)_{L^2} \right| d\tau < \infty.$$

Thus, the right hand side of (36) is well defined. The left hand side is well defined too, because $w(0) = \overset{\circ}{w} \in H$.

In the subsequent sections we will discuss existence and uniqueness of the weak solution.

## 5. EXISTENCE OF THE WEAK SOLUTION

To prove existence of solution of the variational problem (44)-(47) further assumptions about the diffusion coefficients $\tilde{\Theta}_1, \ldots, \tilde{\Theta}_r$ are necessary. These translate into additional properties that the operator $A$ has to satisfy.

From now on we assume that there exist constants $\mu > 0$ and $\vartheta \geq 0$ such that

$$\int_{-\Lambda}^{\Lambda} \left( A_\phi \frac{\partial \psi}{\partial x} \right) \cdot \frac{\partial \psi}{\partial x} \, dx \geq \mu \left\| \frac{\partial \psi}{\partial x} \right\|_{L^2}^2 - \vartheta \, \|\psi\|_{L^2}^2 \tag{49}$$

for any $\phi \in H^1(-\Lambda, \Lambda; \mathbb{R}^r)$, $\phi_1 + \cdots + \phi_r = 1$ and $\psi \in V$. This assumption depends on properties of the map $\tilde{\Theta}$. We present some conditions under which (49) holds with $\vartheta = 0$.

**Remark 5.1.** Assume that for every $\varkappa \in \mathbb{R}^r$ such that $\varkappa_1 + \cdots + \varkappa_r = 1$, $\varkappa_i > 0$, $i = 1, \ldots, r$.

$$\delta_\varkappa + m_\varkappa \sum_{k=1}^{r} \left(\tilde{\Theta}_i(\varkappa) - \delta_\varkappa\right) > 0,$$

where

$$\delta_\varkappa = \min_{i=1,\ldots,r} \tilde{\Theta}_i(\varkappa);$$

$m_\varkappa = 0$ in the case $\left(\tilde{\Theta}_1(\varkappa) = \cdots = \tilde{\Theta}_r(\varkappa)\right)$ or $r = 2$ otherwise

$$m_\varkappa = \min_{s<0} \left\{ s \mid \text{there exists } j \in \{1, \ldots, r\} \mid 4r + \sum_{k \neq j} \frac{\alpha_j^\varkappa + (r-1)\alpha_k^\varkappa - 1}{\alpha_k^\varkappa - s} = 0 \right\}$$

where

$$\alpha_j^\varkappa := \frac{\tilde{\Theta}_j(\varkappa) - \delta_\varkappa}{\sum_{k=1}^{r}\left(\tilde{\Theta}_k(\varkappa) - \delta_\varkappa\right)}.$$

Then the estimation (49) is valid for some constant $\mu \in (0, \infty)$ and $\vartheta = 0$.

The proof of **Remark 5.1** is omitted. **Remark 5.1** states that the operator $\alpha_j^\varkappa$ is strictly positive which is a property common to many operators in mathematical physics. However it is difficult to provide an interpretation of this property in the context of diffusion in the multicomponent mixtures.

We will construct the weak solution of the variational problem (44)-(47) using the Galerkin approximation of the Hilbert space $V$. This method is described elsewhere (Dautray et al., 1988).

**Theorem 5.1.** There exists a weak solution of the variational problem of the interdiffusion in $r$-component solid solution.

*Proof.* We define a family $\{\tilde{e}_j\}_{j=1,\ldots,r-1}$ of vectors in $\mathbb{R}^r$:

$$\tilde{e}_j := \sqrt{\frac{j}{i+1}} \left(\frac{e_1 + \cdots + e_j}{j} - e_{j+1}\right)$$

for $j = 1, \ldots, r-1$. The vectors $\tilde{e}_1, \ldots, \tilde{e}_r$ form an orthonormal basis in $\mathbb{I}^\perp$.

The family of curves:

$$\left\{\frac{1}{\sqrt{\Lambda}} \cos\frac{n\pi}{2\Lambda}(x+\Lambda)\tilde{e}_j\right\}$$

where $n = 1, 2, \ldots$ and $j = 1, \ldots, r-1$, form a Hilbert basis in $H$ and a complete orthogonal system in $V$ with the scalar product $((\cdot|\cdot))$. The sequence of subspaces of the space $V$:

$$X_N = \mathrm{lin}\left\{\frac{1}{\sqrt{\Lambda}} \cos\frac{n\pi}{2\Lambda}(x+\Lambda)\tilde{e}_j\right\},$$

where $n = 1, \ldots, N$, $j = 1, \ldots, r-1$, and $N = 1, 2, \ldots$.

$X_N$ is a Galerkin approximation of the Hilbert space $V$ (Dautray et al., 1988). This family is a Galerkin approximation of $H$ as well. We shall apply **Lemma B.3** with $X = X_N$, see **Appendix B**. Let $\overset{N}{w}$ denote the maximal solution $\mathbf{x}$ from that lemma. The (B.7) implies that for almost every $t \in [0, T]$ and $\phi \in X_N$

$$\left(\frac{d}{dt}\overset{N}{w}(t)|\phi\right)_{L^2} = -\int_{-\Lambda}^{\Lambda}\left(A_{\overset{N}{w}(t)+\mathrm{m}(t)} \frac{\partial \overset{N}{w}(t)}{\partial x}\right) \cdot \frac{\partial \phi}{\partial x}\, dx + K(t)\left(\overset{N}{w}(t)\Big|\frac{\partial\phi}{\partial x}\right)_{L^2} + \phi \cdot \Gamma(t)|_{-\Lambda}^{\Lambda}.$$

and consequently

$$\left(\overset{N}{w}(t)|\phi\right)_{L^2} - \left(\overset{\circ}{w}|\phi\right)_{L^2} = -\int_0^t \int_{-\Lambda}^{\Lambda}\left(A_{\overset{N}{w}(\tau)+\mathrm{m}(\tau)} \frac{\partial \overset{N}{w}(\tau)}{\partial x}\right) \cdot \frac{\partial \phi}{\partial x}\, dx\, d\tau + \int_0^t K(\tau)\left(\overset{N}{w}(\tau)\Big|\frac{\partial\phi}{\partial x}\right)_{L^2} d\tau + \int_0^t \phi \cdot \Gamma(t)\Big|_{-\Lambda}^{\Lambda} d\tau$$

From (B.10) and (B.11) it follows that there exist constants $C$ and $C_1$ such that for any $t \in [0, T]$

$$\left\|\overset{N}{w}(t)\right\|_{L^2}^2 \le C \tag{50}$$

and

$$\int_0^T \left\|\overset{N}{w}(\tau)\right\|_V^2 d\tau \le C_1. \tag{51}$$

Thus the sequence $\left\{\overset{N}{w}\right\}$ is bounded in $L^2(0,\,T;\,V)$. The Banach-Alaoglu theorem (Rudin, 1973) implies that there exists a function $\eta \in L^2(0,\,T;\,V)$ and an infinite subset $\mathcal{N}_1 \subset \mathbb{N}$, such that

$$\overset{N}{w} \to \eta \text{ weakly in } \quad L^2(0,\,T;\,V) \text{ as } \mathcal{N}_1 \ni N \to \infty.$$

The inclusion $\iota : V \hookrightarrow H$ is compact (Adams, 1975) and the operator

$$\pi : H \ni h \mapsto (h|\cdot)_{L^2} \in V'$$

is linear and continuous. A simple computation shows that the sequence $\left\{\pi \circ \iota \circ \overset{N}{w}\right\}_{N=1,\,2,\,\ldots}$ is bounded in $L^2(0,\,T;\,V')$. By the Dubinsky theorem (Dubinskii, 1965) there exist $\zeta \in L^2(0,\,T;\,V)$ and infinite subset $\mathcal{N}_2 \subset \mathcal{N}_1$ such that

$$\iota\circ\overset{N}{w} \to \iota\circ\zeta \quad \text{in} \quad L^2(0,\,T;\,H) \quad \text{as} \quad \mathcal{N}_2 \ni N \to \infty.$$

Since

$$\int_0^T \left(\overset{N}{w}(\tau)\middle|(\eta-\zeta)(\tau)\right)_{L^2} d\tau \to \int_0^T \left(\eta(\tau)\middle|(\eta-\zeta)(\tau)\right)_{L^2} d\tau \text{ as } \mathcal{N}_2 \ni N \to \infty$$

and

$$\int_0^T \left(\overset{N}{w}(\tau)\middle|(\eta-\zeta)(\tau)\right)_{L^2} d\tau \to \int_0^T \left(\zeta(\tau)\middle|(\eta-\zeta)(\tau)\right)_{L^2} d\tau \text{ as } \mathcal{N}_2 \ni N \to \infty$$

the following follows

$$\eta = \zeta.$$

By the Riesz-Fisher theorem (Rudin, 1974) there exists an infinite subset $\mathcal{N}_3 \subset \mathcal{N}_2$ such that for almost every $t \in [0,\,T]$

$$\left\|\overset{N}{w}(t) - \eta(t)\right\|_{L^2} \to 0 \quad \text{as} \quad \mathcal{N}_3 \ni N \to \infty.$$

The formula (49) implies that

$$\|\eta(t)\|^2_{L^2} \le C$$

for almost every $t \in [0,\,T]$.

For $t \in [0,\,T]$, $N \in \mathcal{N}_3$ and $\phi \in V$ we define:

$$\iota_1^N(t)\phi = \int_0^t \int_{-\Lambda}^{\Lambda} \left(A_{\overset{N}{w}(\tau)+\mathrm{m}(\tau)} \frac{\partial \overset{N}{w}(\tau)}{\partial x}\right) \cdot \frac{\partial \phi}{\partial x}\, dx\, d\tau,$$

$$\iota_1(t)\phi = \int_0^t \int_{-\Lambda}^{\Lambda} \left(A_{\eta(\tau)+\mathrm{m}(\tau)} \frac{\partial \eta(\tau)}{\partial x}\right) \cdot \frac{\partial \phi}{\partial x}\, dx\, d\tau,$$

$$\iota_2^N(t)\phi = \int_0^t K(\tau) \left(\overset{N}{w}(\tau)\middle|\frac{\partial \phi}{\partial x}\right)_{L^2} d\tau,$$

$$\iota_2(t)\phi = \int_0^t K(\tau) \left(\eta(\tau)\middle|\frac{\partial \phi}{\partial x}\right)_{L^2} d\tau.$$

After elementary computations we get:

$$\left.\begin{aligned} \iota_1^N(t)\phi &\longrightarrow \iota_1(t)\phi \\ \iota_2^N(t)\phi &\longrightarrow \iota_2(t)\phi \end{aligned}\right\} \text{ as } \mathcal{N}_3 \ni N \to \infty. \tag{52}$$

The functionals $\iota_1^N,\ \iota_1,\ \iota_2^N,\ \iota_2 : V \to \mathbb{R}$ are linear and continuous. Moreover,

$$\left|\iota_1^N(t)\right|_{V'} \le \|A\|_{L^\infty} \sqrt{TC_1}, \tag{53}$$

$$\left|\iota_2^N(t)\right|_{V'} \le \|K\|_{L^2} \sqrt{TC}$$

for any $t \in [0,\,T]$ and $N \in \mathcal{N}_3$.

Let $P_N : L^2(-\Lambda,\,\Lambda;\,\mathbb{R}^r) \to X_N$ be the orthogonal projection. Fix $t \in [0,\,T]$ and $\phi \in V$. If $\phi_N = P_N\,\phi$, then

$$\left(\overset{N}{w}(t)\middle|\phi\right)_{L^2} = \left(\overset{N}{w}(t)\middle|\phi_N\right)_{L^2} = \left(\overset{\circ}{w}\middle|\phi_N\right)_{L^2} - \iota_1^N(t)\phi_N + \iota_2^N(t)\phi_N + \int_0^t \phi\cdot\Gamma(\tau)\Big|_{-\Lambda}^{\Lambda} d\tau.$$

Since $\phi_N \to \phi$ in $V$ for $N \to \infty$, therefore $\phi_N \to \phi$ in $L^\infty(-\Lambda,\,\Lambda;\,\mathbb{R}^r)$. Thus

$$\left(\overset{N}{w}(t)\middle|\phi\right)_{L^2} \longrightarrow \left(\overset{\circ}{w}\middle|\phi\right)_{L^2} - \iota_1(t)\phi + \iota_2(t)\phi + \int_0^t \phi\cdot\Gamma(\tau)\Big|_{-\Lambda}^{\Lambda} d\tau$$

for $\mathcal{N}_3 \ni N \to \infty$. It means that the sequence $\left\{\overset{N}{w}(t)\right\}_{N\in\mathcal{N}_3}$ is weakly convergent in $V$. Because $V$ is dense in $H$, the sequence $\left\{\overset{N}{w}(t)\right\}_{N\in\mathcal{N}_3}$ is weakly convergent in $H$ either. Consequently, there is a unique vector $w(t) \in H$ such that for every $\phi \in H$

$$w : [0,\,T] \to H.$$

It is obvious that for any $t \in [0,\,T]$ and $\phi \in V$

$$\left(w(t)\middle|\phi\right)_{L^2} = \left(\overset{\circ}{w}\middle|\phi\right)_{L^2} - \iota_1(t)\phi + \iota_2(t)\phi + \int_0^t \phi\cdot\Gamma(\tau)\Big|_{-\Lambda}^{\Lambda} d\tau.$$

In particular $w(0) = \overset{\circ}{w}$.

By the Mazur theorem (Megginson, 1998) and (49)

$$\|w(t)\|^2_{L^2} \le C.$$

for any $t \in [0,\,T]$. Let $t_* \in [0,\,T]$ and $t_k \longrightarrow t_*$ as $k \to \infty$. Since the functions $\iota_1$ and $\iota_2$ are continuous we compute that

$$\lim_{k\to\infty} \left(w(t_k)\middle|\phi\right)_{L^2} = \left(w(t_*)\middle|\phi\right)_{L^2}$$

for any $\phi \in V$. Since $V$ is dense in $H$, the curve $w : [0, T] \to H$ is weakly continuous and the condition (44) of **Definition 4.1** is satisfied.
By the definition of $w$ and $\eta$

$$w(t) = \eta(t) \tag{54}$$

for almost every $t \in [0, T]$, and consequently (45) and (46) hold. Thus, from (52) and (54)

$$\begin{aligned}(w(t)|\phi)_{L^2} = &\\ &\left(\overset{\circ}{w}|\phi\right)_{L^2} - \\ &\int_0^t \int_{-\Lambda}^{\Lambda} \left(A_{w(\tau)+\mathrm{m}(\tau)} \frac{\partial w(\tau)}{\partial x}\right) \frac{\partial \phi}{\partial x}\, dx\, d\tau + \\ &\int_0^t K(\tau) \left(w(\tau) \Big| \frac{\partial \phi}{\partial x}\right)_{L^2} d\tau + \\ &\int_0^t \phi \cdot \Gamma(\tau)\Big|_{-\Lambda}^{\Lambda} d\tau\end{aligned} \tag{55}$$

which means that (47) is true. This completes the proof. □

From (44), every weak solution $w : [0, T] \to H$ of the variational problem is weakly continuous. Below we show that $w$ is actually continuous.

**Proposition 5.1.** Every weak solution of the variational problem (44)-(47) $w : [0, T] \to H$ is a continuous function. For any $t \in [0, T]$:

$$\begin{aligned}&\frac{1}{2}\left(\|w(t)\|^2_{L^2} - \left\|\overset{\circ}{w}\right\|^2_{L^2}\right) = \\ &\quad -\int_0^t \int_{-\Lambda}^{\Lambda} \left(A_{w(\tau)+\mathrm{m}(\tau)} \frac{\partial w(\tau)}{\partial x}\right) \cdot \frac{\partial w}{\partial x}\, dx\, d\tau + \\ &\int_0^t K(\tau) \left(w(\tau) \Big| \frac{\partial w(\tau}{\partial x}\right)_{L^2} d\tau + \\ &\int_0^t w(\tau) \cdot \Gamma(\tau)\Big|_{-\Lambda}^{\Lambda} d\tau.\end{aligned}$$

*Proof.* For $\tau \in [0, T]$ such that $w(\tau) \in V$ and $\phi \in V$ we define:

$$\begin{aligned}f_1(\tau)(\phi) &= -\int_{-\Lambda}^{\Lambda} \left(A_{w(\tau)+\mathrm{m}(\tau)} \frac{\partial w(\tau)}{\partial x}\right) \cdot \frac{\partial w}{\partial x}\, dx, \\ f_2(\tau)(\phi) &= K(\tau) \left(w(\tau) \Big| \frac{\partial \phi}{\partial x}\right)_{L^2}, \\ f_3(\tau)(\phi) &= \phi \cdot \Gamma(\tau)\Big|_{-\Lambda}^{\Lambda}.\end{aligned}$$

One can check that for almost every $\tau \in [0, T]$ $f_1(\tau), f_2(\tau), f_3(\tau) \in V'$ and

$$\begin{aligned}\|f_1(\tau)\|_{V'} &\le \|A\|_{L^\infty} \|w(\tau)\|_V, \\ \|f_2(\tau)\|_{V'} &\le |K(\tau)| \|w(\tau)\|_V, \\ \|f_3(\tau)\|_{V'} &\le \sqrt{\frac{2\Lambda}{3}} (|\Gamma_R(\tau)| + |\Gamma_L(\tau)|).\end{aligned}$$

Consequently

$$f = f_1 + f_2 + f_3 \in L_2(0, T; V'). \tag{56}$$

By (55), for any $t \in [0, T]$ and $\phi \in V$

$$(w(t)|\phi)_{L^2} = \left(\overset{\circ}{w}|\phi\right)_{L^2} + \int_0^t f(\tau)(\phi)\, d\tau. \tag{57}$$

It follows that for almost every $t \in [0, T]$

$$\frac{d}{dt}(w(t)|\phi)_{L^2} = f(\tau)(\phi).$$

Let $\tau \in w^{-1}(V)$ such that (57) holds and let $t_k \longrightarrow \tau$ as $k \to \infty$. By (57), the sequence $\frac{w(t_k)-w(\tau)}{t_k-\tau}$ is weakly bounded. Thus it is bounded in the norm $\|\cdot\|_{L^2}$. This implies that $w$ is continuous in $\tau$ and

$$\begin{aligned}f(\tau)(w(\tau)) = &\\ &\frac{d}{dt}(w(t)|w(\tau))_{L^2}\Big|_{t=\tau} = \\ &\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2}\Big|_{t=\tau}.\end{aligned} \tag{58}$$

Consequently, for almost every $\tau \in [0, T]$ $w$ is continuous. Since $w$ is weakly continuous then $w$ is continuous for any $\tau \in [0, T]$.

Integrating (58) we get:

$$\frac{1}{2}\left(\|w(t)\|^2_{L^2} - \left\|\overset{\circ}{w}\right\|^2_{L^2}\right) = \int_0^t f(\tau)(w(\tau))\, d\tau$$

for any $t \in [0, T]$ and the proof is completed. □

## 6. UNIQUENESS OF THE WEAK SOLUTION

**Theorem 5.1** guarantees that at least one weak solution $w : [0, T] \to H$ of the variational problem (44)-(47) exists. However, to show that it is unique we will need to strenghten (46), namely

$$\int_0^T \|w(\tau)\|^4_V\, d\tau < \infty. \tag{59}$$

**Theorem 6.1.** Let $w$ and $\bar{w}$ be weak solutions of the variational problem (44)-(47). If $w$ satisfies (59), then

$$w = \bar{w}.$$

*Proof.* For almost all $\tau \in [0, T]$ and any $\phi \in V$ define $\bar{f}$ in similar manner as $f$:

$$\bar{f}(\tau)(\phi) = -\int_{-\Lambda}^{\Lambda} \left( A_{\bar{w}(\tau)+\mathrm{m}(\tau)} \frac{\partial \bar{w}(\tau)}{\partial x} \right) \cdot \frac{\partial \phi}{\partial x}\, dx + K(\tau) \left( \bar{w}(\tau) \Big| \frac{\partial \phi}{\partial x} \right)_{L^2} + \phi \cdot \Gamma(\tau) \Big|_{-\Lambda}^{\Lambda}$$

As for $f$, it is evident that $\bar{f} \in L^2(0, T; V')$ and for $t \in [0, T]$ and $\phi \in V$

$$\left(\bar{w}(t) | \phi\right)_{L^2} = \left(\mathring{w} | \phi\right)_{L^2} + \int_0^t \bar{f}(\tau)(\phi)\, d\tau. \quad (60)$$

Since for almost every $t \in [0, T]$, $\bar{w}(t) \in V'$ then we can replace $\phi$ in the formulae (57) and (60) by $\bar{w}(t)$. Then for almost every $t \in [0, T]$

$$\|\bar{w}(t)\|_{L^2}^2 - \left(w(t) | \bar{w}(t)\right)_{L^2} = \int_0^t \left(\bar{f}(\tau) - f(\tau)\right) \left(\bar{w}(\tau)\right) d\tau. \quad (61)$$

It is obvious that the same formula is valid if we swap $\bar{w}$ and $w$, $\bar{f}$ and $f$. Consequently, for almost every $t \in [0, T]$

$$\|w(t)\|_{L^2}^2 - \left(w(t) | \bar{w}(t)\right)_{L^2} = \int_0^t \left(f(\tau) - \bar{f}(\tau)\right) \left(w(\tau)\right) d\tau \quad (62)$$

Upon adding (61) and (62) we get

$$\|w(t) - \bar{w}(t)\|_{L^2}^2 = \int_0^t \left(\bar{f}(\tau) - f(\tau)\right) \left(\bar{w}(\tau)\right) + \left(f(\tau) - \bar{f}(\tau)\right) \left(w(\tau)\right) d\tau.$$

Let us denote $\bar{\bar{w}} = \bar{w} - w$. After simple computations we obtain:

$$\frac{1}{2} \|\bar{\bar{w}}(t)\|^2 = -\int_0^t \int_{-\Lambda}^{\Lambda} \left( A_{\bar{w}(\tau)+\mathrm{m}(\tau)} \frac{\partial \bar{\bar{w}}(\tau)}{\partial x} \right) dx\, d\tau + - \int_0^t \int_{-\Lambda}^{\Lambda} \left( A_{\bar{w}(\tau)+\mathrm{m}(\tau)} - A_{w(\tau)+\mathrm{m}(\tau)} \right) \left( \frac{\partial w(\tau)}{\partial x} \cdot \frac{\partial \bar{\bar{w}}(\tau)}{\partial x} \right) dx\, d\tau + \int_0^t K(\tau) \left( \bar{\bar{w}}(\tau) \Big| \frac{\partial \bar{\bar{w}}(\tau)}{\partial x} \right)_{L^2} d\tau$$

for almost every $t \in [0, T]$. Below we will estimate each term.

By (49) for $\psi = \bar{\bar{w}}$ we obtain

$$\int_{-\Lambda}^{\Lambda} \left( A_{\bar{w}(\tau)+\mathrm{m}(\tau)} \frac{\partial \bar{\bar{w}}(\tau)}{\partial x} \right) \cdot \frac{\partial \bar{\bar{w}}(\tau)}{\partial x}\, dx \geq \mu \left\| \frac{\partial \bar{\bar{w}}(\tau)}{\partial x} \right\|_{L^2}^2 - \vartheta \|\bar{\bar{w}}(\tau)\|_{L^2}^2. \quad (63)$$

For $x \in [-\Lambda, \Lambda]$ and $\tau \in [0, T]$, **Lemma A.2** implies

$$\left| A_{\bar{w}(\tau)+\mathrm{m}(\tau)} - A_{w(\tau)+\mathrm{m}(\tau)} \right| \leq L_A |\bar{w}(\tau, x) - w(\tau, x)| \leq L_A \|\bar{\bar{w}}(\tau)\|_{L^\infty}.$$

Since for almost every $\tau \in [0, T]$

$$\|\bar{\bar{w}}(\tau)\|_{L^\infty} \leq \sqrt{2} \|\bar{\bar{w}}(\tau)\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \bar{\bar{w}}(\tau)}{\partial x} \right\|_{L^2}^{\frac{1}{2}},$$

then, by the Hölder's inequality:

$$\int_{-\Lambda}^{\Lambda} \left| A_{\bar{w}(\tau)+\mathrm{m}(\tau)} - A_{w(\tau)+\mathrm{m}(\tau)} \right| \left| \frac{\partial w(\tau)}{\partial x} \right| \left\| \frac{\partial \bar{\bar{w}}(\tau)}{\partial x} \right\| dx \leq \leq \sqrt{2} L_A \|\bar{\bar{w}}(\tau)\|_{L^2}^{\frac{1}{2}} \|w(\tau)\|_V \|\bar{\bar{w}}(\tau)\|_V^{\frac{3}{2}}. \quad (64)$$

Applying the Young's inequality with 4 and $\frac{4}{3}$ we get for any $\delta > 0$

$$\sqrt{2} L_A \|\bar{\bar{w}}(\tau)\|_{L^2}^{\frac{1}{2}} \|w(\tau)\|_V \|\bar{\bar{w}}(\tau)\|_V^{\frac{3}{2}} \leq \leq \left( \frac{1}{\delta} L_A \right)^4 \|\bar{w}(\tau)\|_V^4 \|\bar{\bar{w}}(\tau)\|_{L^2}^2 + \frac{3}{4} \delta^{\frac{4}{3}} \|w(\tau)\|_V^2. \quad (65)$$

It remains to estimate

$$\left| K(\tau) \left( \bar{\bar{w}}(\tau) \Big| \frac{\partial \bar{\bar{w}}(\tau)}{\partial x} \right)_{L^2} \right| \leq |K(\tau)| \|\bar{\bar{w}}(\tau)\|_{L^2} \|\bar{\bar{w}}(\tau)\|_V \leq \frac{1}{2\delta^2} K(\tau)^2 \|\bar{\bar{w}}(\tau)\|_{L^2}^2 + \frac{1}{2} \delta^2 \|\bar{\bar{w}}(\tau)\|_V^2 \quad (66)$$

for any $\delta > 0$. Since the function: $(0, \infty) \ni \delta \mapsto \frac{3}{4}\delta^{\frac{4}{3}} + \frac{1}{2}\delta^2 \in (0, \infty)$ is continuous then there exists $\delta_* \in (0, \infty)$, such that

$$\frac{3}{4} \delta_*^{\frac{4}{3}} + \frac{1}{2} \delta_*^2 = \mu.$$

Upon combining (63), (64) and (66), we finally obtain

$$\|\bar{\bar{w}}(t)\|_{L^2}^2 \leq \int_0^t 2 \left( \vartheta + \left( \frac{L_A}{\delta_*} \right)^4 \|w(t)\|_V^4 + \frac{1}{2\delta_*^2} K(\tau)^2 \right) \|\bar{\bar{w}}(\tau)\|_{L^2}^2\, d\tau \quad (67)$$

for almost every $t \in [0, T]$. Since $\bar{\bar{w}} : [0, T] \to H$ is a continuous function the inequality (67) holds for every $t \in [0, T)$. The Gronwall's inequality implies that for almost every $t \in [0, T)$

$$\|\bar{\bar{w}}(t)\|_{L^2}^2 = 0$$

and the continuity of $\bar{\bar{w}}$ results in

$$\bar{\bar{w}} \equiv 0.$$

Consequently, $w = \bar{w}$ and the proof is completed. □

## 7. THE PROBLEM OF INTERDIFFUSION FOR TIMES $0 \leq t < \infty$

We formulated the boundary value problem of interdiffusion in the $r$-component one dimensional solid solution so that the examination time was finite $T < \infty$. In this section we examine the situation when $T = \infty$. The classical boundary value problem in **Section 3** requires only replacement of the compact interval $[0, T]$ by the unbound interval $[0, \infty)$ and all assumptions about the data and unknowns will remain intact. Such a substitution would be too restrictive for the variational problem (44)-(47).

**Definition 7.1.** A function $w : [0, \infty) \to H$ is a weak solution of the boundry value problem of interdiffusion in the $r$-component one dimensional solid solution if

$$w \text{ is weakly continuous;} \tag{68}$$

$$\text{for almost all} \quad t \in [0, \infty) \; : \; w(t) \in V; \tag{69}$$

$$\text{for any} \quad T > 0 \quad \int_0^T \|w(t)\|_V^2 \, dt < \infty; \tag{70}$$

$$\text{for any} \quad t \geq 0 \quad \text{and} \quad \phi \in V \quad \text{(36) holds.} \tag{71}$$

Existence of a solution the problem (68)-(71) can be concluded from the proof of **Theorem 5.1**.

**Theorem 7.1.** Let the assumptions for **Theorem 5.1** hold. Then there exists a solution of the variational problem (68)-(71).

*Proof.* For a fixed $T > 0$ let $w_T : [0, T] \to H$ denote the solution constructed in the proof of **Theorem 5.1**. If $T_1 < T_2$, then for $t \in [0, T]$

$$w_{T_1}(t) = w_{T_2}(t)$$

since $w_{T_1}$ and $w_{T_2}$ are solutions of the problem (**Lemma B.3**) with the same initial condition. Consequently, the function

$$w = \bigcup_{T>0} w_T$$

is well defined for $t > 0$. Veryfying that $w$ satisfies (68)-(71) is straight forward. □

If $w : [0, \infty) \to H$ is a weak solution and $T > 0$, then $w|_{[0,T]}$ is a solution of the variational problem (44)-(47). Hence $w$ satisfies **Proposition 5.1**. Also, if (59) holds for any $T > 0$ , then $w$ is a unique solution of the variational problem (68)-(71).

Suppose that the constant $\vartheta$ from the Garding's inequality (49) equals zero. For the closed system, there is no flux of mass through the boundaries which means that $K \equiv 0$ , $\Gamma_R = \Gamma_L = 0$. Then the following asymptotic behavior of $\|w\|_{L^2}^2$ for large times can be determined.

**Theorem 7.2.** Let $w : [0, \infty) \to H$ be a weak solution of the variational problem (68)-(71) with $K \equiv 0$, $\Gamma_R = 0$ and $\Gamma_L = 0$. Assume additionally that

$$\int_{-\Lambda}^{\Lambda} \left( A_{w(\tau)+\mathrm{m}(\tau)} \frac{\partial w(\tau)}{\partial x} \right) \cdot \frac{\partial w(\tau)}{\partial x} \, dx \geq \mu \int_{-\Lambda}^{\Lambda} \left| \frac{\partial w(\tau)}{\partial x} \right|^2 dx \tag{72}$$

for almost every $\tau \in [0, \infty)$. Then $w : [0, \infty) \to H$ is continuous and bounded,

$$\int_0^\infty \left\| \frac{\partial w(\tau)}{\partial x} \right\|_{L^2}^2 d\tau < \infty,$$

the function $[0, \infty) \ni t \mapsto \|w(t)\|_{L^2}^2 \in [0, \infty)$ is decreasing and

$$\lim_{t\to\infty} \|w(t)\|_{L^2}^2 = 0.$$

*Proof.* For the closed system $K \equiv 0$ , $\Gamma_R = 0$ and $\Gamma_L = 0$. **Proposition 5.1** implies that $w : [0, \infty) \to H$ is continuous and

$$\frac{1}{2}\left( \|w(t)\|_{L^2}^2 - \left\| \overset{\circ}{w} \right\|_{L^2}^2 \right) = - \int_0^t \int_{-\Lambda}^{\Lambda} \left( A_{w(\tau)+\mathrm{m}(\tau)} \frac{\partial w(\tau)}{\partial x} \right) \cdot \frac{\partial w(\tau)}{\partial x} \, dx \, d\tau \tag{73}$$

for any $t \in [0, \infty)$. Applying the assumption (72) we can estimate for any $t \in [0, \infty)$:

$$\|w(t)\|_{L^2}^2 + 2\mu \int_0^t \|w(\tau)\|_V^2 \, d\tau \leq \left\| \overset{\circ}{w} \right\|_{L^2}^2 .$$

Hence

$$\sup_{t\in[0,\infty)} \|w(t)\|_{L^2}^2 \leq \left\| \overset{\circ}{w} \right\|_{L^2}^2$$

and
$$\int_0^\infty \|w(\tau)\|_V^2 \, d\tau \le \left\| \overset{\circ}{w} \right\|_{L^2}^2 . \tag{74}$$
Upon differentiation (73) and taking into account the assumption (72) we get for almost every $t \in [0, \infty)$
$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 \le -\mu \left\| \frac{\partial w(\tau)}{\partial x} \right\|_{L^2}^2 .$$
Thus $[0, \infty) \ni t \mapsto \|w(t)\|_{L^2}^2 \in [0, \infty)$ is a decreasing function. The inequality (74) implies that there exists a sequence $\{t_k\}_{k \in \mathbb{N}}$ such that $t_k \to \infty$ and $\|w(t_k)\|_V^2 \to 0$ as $k \to \infty$. Consequently, $w(t_k) \to 0$ in $V$ and also $w(t_k) \to 0$ in $H$. It means that
$$\lim_{t \to \infty} \|w(t)\|_{L^2}^2 = 0.$$
That is the end of the proof. □

The above theorem has a fundamental interpretation. Since $w$ is a vector representing the local deviation of the molar ratio of the mixture components, consequently the above integral represents the random deviation (variance) of molar ratio of the mixture components. Thus, **Theorem 7.2** asserts that standard deviation of molar ratio of mixture components in the closed system is monotonically decreasing to zero when time $t \to \infty$. In still simpler words it says that in the closed system and for the long observation time, the mixture becomes homogeneous. Accordingly the mathematical model has the property of a real mixture.

## 8. SUMMARY

1. The chemical interdiffusion problem in one-dimensional $r$-component mixture (in which the diffusivities of every component can be a function of composition) has been formulated. The model bases on the concept of Darken's flow, the law of mass conservation and on the retarded form of equation of state (i.e. on the assumption of the constant molar volume of the mixture).
2. The mathematical model results in the formulation of the variational form of interdiffusion problem.
3. The proof of existence and uniqueness of a weak solution such problem is presented.
4. The asymptotic behavior of the weak solution when $t \to \infty$ was investigated.
5. It has been proved that when the mixture is in the closed system, the gradients of density of every mixture component vanish at the mixture boundaries.

## A. APPENDIX

Below we prove some properties of the map $A$ that are necessary for analysis of the variational problem.

**Lemma A.1.** The mapping
$$A : \{\varkappa \in \mathbb{R}_+^r \mid \varkappa_1 + \cdots + \varkappa_r = 1\} \ni \varkappa \mapsto A_\varkappa,$$
where $A_\varkappa \in \mathrm{End}\left(\mathbb{I}^\perp\right)$, is bounded and satisfies the Lipschitz condition. $\mathrm{End}\left(\mathbb{I}^\perp\right)$ is the space of all endomorphisms of $\mathbb{I}^\perp$ with the standard norm.

*Proof.* By (24) the mappings $\tilde{\Theta}_1, \ldots, \tilde{\Theta}_r$ satisfy the Lipschitz condition. Thus, there are constants $L_1, \ldots, L_r > 0$ such that for $\varkappa_1, \varkappa_2 \in \mathrm{dom}\, A$
$$\left|\tilde{\Theta}_i(\varkappa_1) - \tilde{\Theta}_i(\varkappa_2)\right| \le L_i \left|\varkappa_1 - \varkappa_2\right| .$$
Let $\xi \in \mathbb{I}^\perp$. One can easily estimate
$$|A_{\varkappa_1}(\xi) - A_{\varkappa_2}(\xi)| \le \left(\left(\sum_{i=1}^r L_i^2\right)^{\frac{1}{2}} + \sum_{i=1}^r L_i\right) |\varkappa_1 - \varkappa_2|\, |\xi| .$$
Therefore
$$|A_{\varkappa_1}(\xi) - A_{\varkappa_2}(\xi)| \le L_A \left|\varkappa_1 - \varkappa_2\right| ,$$
where $L_A = \left(\sum_{i=1}^r L_i^2\right)^{\frac{1}{2}} + \sum_{i=1}^r L_i$. Since for $\varkappa \in \mathrm{dom}\, A$
$$|A_\varkappa| \le 2\left|\tilde{\Theta}(\varkappa)\right| ,$$
and the mapping $\tilde{\Theta}$ is bounded, so is the mapping $A$. That completes the proof. □

**Lemma A.2.** The mapping
$$(\phi, \psi) \mapsto A_\psi \frac{\partial \psi}{\partial x} \in L^2\left(-\Lambda, \Lambda; \mathbb{I}^\perp\right) \tag{A.1}$$
where $(\phi, \psi) \in \{\phi \in H^1(-\Lambda, \Lambda; \mathbb{R}^r) \mid \phi_1 + \cdots + \phi_r = 1\} \times V$, satisfies the Lipschitz condition on every bounded set in $H^1(-\Lambda, \Lambda; \mathbb{R}^r) \times H^1(-\Lambda, \Lambda; \mathbb{R}^r)$ and
$$\left\| A_\phi \frac{\partial \psi}{\partial x} \right\|_{L^2} \le \|A\|_{L^\infty} \left\| \frac{\partial \psi}{\partial x} \right\|_{L^2} .$$

*Proof.* Since the inclusion

$$j : H^1(-\Lambda, \Lambda; \mathbb{R}^r) \hookrightarrow L^\infty(-\Lambda, \Lambda; \mathbb{R}^r)$$

is continuous (Adams, 1975), one can estimate

$$\left\| A_{\phi_1} \frac{\partial \psi_1}{\partial x} - A_{\phi_2} \frac{\partial \psi_2}{\partial x} \right\|_{L^2} \leq L_A \left\| \phi_1 - \phi_2 \right\|_{L^\infty} \left\| \frac{\partial \psi_1}{\partial x} \right\|_{L^2} + \left\| A \right\|_{L^\infty} \left\| \frac{\partial (\psi_1 - \psi_2)}{\partial x} \right\|_{L^2} \leq \max \left\{ L_A |j| \left\| \psi_1 \right\|_{H^1}, \left\| A \right\|_{L^\infty} \right\} \left( \left\| \phi_1 - \phi_2 \right\|_{H^1} + \left\| \psi_1 - \psi_2 \right\|_{H^1} \right)$$

for $(\phi_1, \psi_1)$, $(\phi_2, \psi_2)$ in the domain of the function (A.1). Thus the mapping is Lipschitz on every bounded set in $H^1(-\Lambda, \Lambda; \mathbb{R}^r) \times H^1(-\Lambda, \Lambda; \mathbb{R}^r)$. If we take $\phi_1 = \phi_2$ and $\psi_1 =$ **const** in the above estimation we get

$$\left\| A_{\phi_2} \frac{\partial \psi_2}{\partial x} \right\|_{L^2} \leq \left\| A \right\|_{L^\infty} \left\| \frac{\partial \psi_2}{\partial x} \right\|_{L^2}.$$

That is the end of the proof. □

## B. APPENDIX

In this appendix we derive a priori estimations of solution of the approximate problems in the finite dimensional subspaces of $V$ that form a Galerkin approximation of $V$ (Dautray et al., 1988).

**Definition B.1.** Let $X$ be a Banach space. Consider the problem

$$\dot{x} = F(t, x), \tag{B.1}$$

where $F : \mathbb{R} \times X \multimap X$. The symbol $\multimap$ means that $\operatorname{dom} F \subset \mathbb{R} \times X$. An absolutely continuous curve $x : \mathbb{R} \multimap X$ is a solution of the problem (B.1) if and only if $\operatorname{dom} w$ is a connected subset of $\mathbb{R}$ and for almost every $t \in \operatorname{dom} x$

$$\dot{x}(t) = F(t, x(t)).$$

The problem (B.1) is solvable if for any $(t_0, x_0) \in \operatorname{dom} F$ there is a solution of this problem such that:

$$x(t_0) = x_0. \tag{B.2}$$

The problem (B.1) is uniquely solvable if for any $(t_0, x_0) \in \operatorname{dom} F$ the problem: (B.1), (B.2) has the unique solution. The above definition can be used to formulate the Picard's type theorem.

**Theorem B.1.** Assume that the map $F$ has an open domain and satisfies locally the Lipschitz condition with respect to the second variable. If for any continuous function $\gamma : \mathbb{R} \multimap X$ such that $\gamma \subset \operatorname{dom} F$, the curve $t \mapsto F(t, \gamma(t))$ is locally summable, then the problem (B.1) is solvable.

The proof of **Theorem B.1** is based on the proof of the Picard theorem (Coddington et al., 1955).

**Lemma B.1.** Let $X \subset V$ be a subspace of $H$ which has a finite dimension. For $\phi \in L^2(-\Lambda, \Lambda; \mathbb{R}^r)$ we denote by $B(\phi)$ the only vector in $X$ such that for any $\psi \in X$

$$\int_{-\Lambda}^{\Lambda} \phi \cdot \frac{\partial \psi}{\partial x} \, dx = \int_{-\Lambda}^{\Lambda} B(\phi) \cdot \psi \, dx. \tag{B.3}$$

For $\gamma = (\gamma_L, \gamma_R) \in \mathbb{R}^r \times \mathbb{R}^r$ we denote by $G(\gamma)$ the only vector in $X$ such that for any $\psi \in X$

$$\psi(\Lambda)\gamma_R - \psi(-\Lambda)\gamma_L = \int_{-\Lambda}^{\Lambda} G(\gamma) \cdot \psi \, dx. \tag{B.4}$$

Then

**1.** The differential equation:

$$\dot{\mathbf{x}}(t) = -B\left( A_{\mathbf{x}(t)+\mathrm{m}(t)} \frac{\partial \mathbf{x}(t)}{\partial x} \right) + K(t)\, B(\mathbf{x}(t)) + G(\Gamma(t)) \tag{B.5}$$

has an unique solution;

**2.** Every maximal solution $\mathbf{x}$ of the equation (B.5) such that

$$\mathbf{x}(0) = P_X \overset{\circ}{w}, \tag{B.6}$$

where $P_X$ is the orthogonal projection of $L^2(-\Lambda, \Lambda; \mathbb{R}^r)$ onto $X$, has the domain $\operatorname{dom} \mathbf{x} = [0, T)$.

Moreover

(a) for almost every $t \in [0, T)$ and any $\phi \in X$

$$\left( \dot{\mathbf{x}}(t) \middle| \phi \right)_{L^2} = -\int_{-\Lambda}^{\Lambda} \left( A_{\mathbf{x}(t)+\mathrm{m}(t)} \frac{\partial \mathbf{x}(t)}{\partial x} \right) \cdot \frac{\partial \phi}{\partial x} \, dx + K(t) \left( \mathbf{x}(t) \middle| \frac{\partial \phi}{\partial x} \right)_{L^2} + \Gamma_R(t) \cdot \phi(\Lambda) - \Gamma_L(t) \cdot \phi(-\Lambda), \tag{B.7}$$

(b) for any $t \in [0, T)$

$$\int_0^t \left\| \mathbf{x}(\tau) \right\|_V^2 \, d\tau \leq C_1 \tag{B.8}$$

and

$$\left\| \mathbf{x}(t) \right\|_{L^2}^2 \leq C \tag{B.9}$$

where

$$C = \left( \|\mathring{w}\|_{L^2}^2 + \frac{2\Lambda}{3\mu} \int_0^T |\Gamma(\tau)| \, d\tau \right) \exp\left( 2\vartheta T + \frac{1}{\mu} \int_0^T K(\tau)^2 \, d\tau \right) \tag{B.10}$$

and

$$C_1 = \frac{1}{\mu} \left( \left\|\mathring{w}\right\|_{L^2}^2 + \frac{4\Lambda}{3\mu} \int_0^T |\Gamma(\tau)|^2 \, d\tau \right) + 2C\frac{1}{\mu} \left( \vartheta T + \frac{1}{\mu} \int_0^T K(\tau)^2 \, d\tau \right). \tag{B.11}$$

*Proof.* We apply **Theorem B.1**. Here the map $F$ has the form

$$F(t, \psi) = F_1(t, \psi) + F_2(t, \psi) \tag{B.12}$$

for $(t, \psi) \in [0, T) \times X$, where

$$F_1(t, \psi) = -B \left( A_{\psi + \mathrm{m}(t)} \frac{\partial \psi}{\partial x} \right)$$

and

$$F_2(t, \psi) = K(t) B(\psi) + G(\Gamma(t)).$$

Since the map (A.1) satisfies Lipschitz condition on every bounded set, then one can easily check that $F_1$ is a locally Lipschitz map with respect to the second variable. Let $t \in (0, T)$, $\psi_1 \in X \ni \psi_2$. Estimate:

$$\begin{aligned} \|F_2(t, \psi_L) - F_2(t, \psi_R)\|_V \leq \\ |K(t)| \, \|B(\psi_1 - \psi_2)\|_V \leq \\ |K(t)| \, \|B\| \, \|\psi_1 - \psi_2\|_{L^2}, \end{aligned}$$

where we apply the fact that the operator

$$B : L^2(-\Lambda, \Lambda; \mathbb{R}^r) \to X$$

is linear and continuous. Since the inclusion

$$V \hookrightarrow L^2(-\Lambda, \Lambda; \mathbb{R}^r)$$

is continuous, it follows that $F_2$ is a Lipschitz function with respect to the second variable. Consequently, $F$ locally satisfies the Lipschitz condition.

Let $\gamma : \mathbb{R} \multimap X$ be a continuous funtion. The curve $t \mapsto F_1(t, \gamma(t))$ is continuous. Therefore it is locally summable. Since the functions $K$ and $\Gamma$ are locally summable (see (38)-(40)), it follows that the curve $t \mapsto F_2(t, \gamma(t))$ is locally summable as well. Thus $F$ satisfies all assumptions of **Theorem B.1**. Consequently, the proof of (a) is complete.

Let $\mathbf{x} : \mathbb{R} \multimap X$ be a maximal solution of the problem (B.5) such that $\mathbf{x}(0) = P_X \mathring{w}$. Since $\operatorname{dom} F = [0, T) \times X$, then $\operatorname{dom} \mathbf{x}$ is open in $[0, T)$. Let $\tau \in \operatorname{dom} \mathbf{x}$, multiplying both sides of the equation (B.5) by $\phi$ we get:

$$\begin{aligned} \left(\dot{\mathbf{x}}(\tau) \middle| \phi\right)_{L^2} = \\ - \left( A_{\mathbf{x}(\tau)+\mathrm{m}(\tau)} \frac{\partial \mathbf{x}(\tau)}{\partial x} \middle| \frac{\partial \phi}{\partial x} \right)_{L^2} + \\ K(\tau) \left( \mathbf{x}(\tau) \middle| \frac{\partial \phi}{\partial x} \right)_{L^2} + \\ \phi(\Lambda) \cdot \Gamma_R(\tau) - \phi(-\Lambda) \cdot \Gamma_L(\tau). \end{aligned} \tag{B.13}$$

Upon substituting $\phi = \mathbf{x}(t)$ we get:

$$\begin{aligned} \left(\dot{\mathbf{x}}(\tau) \middle| \mathbf{x}(\tau)\right)_{L^2} = \\ - \left( A_{\mathbf{x}(\tau)+\mathrm{m}(\tau)} \frac{\partial \mathbf{x}(\tau)}{\partial x} \middle| \frac{\partial \mathbf{x}(\tau)}{\partial x} \right)_{L^2} + \\ K(\tau) \left( \mathbf{x}(\tau) \middle| \frac{\partial \mathbf{x}(\tau)}{\partial x} \right)_{L^2} + \\ \mathbf{x}(\tau)(\Lambda) \cdot \Gamma_R(\tau) - \mathbf{x}(\tau)(-\Lambda) \cdot \Gamma_L(\tau). \end{aligned} \tag{B.14}$$

Consequently,

$$\begin{aligned} \frac{1}{2} \frac{d}{d\tau} \|\mathbf{x}(\tau)\|_{L^2}^2 = \left(\dot{\mathbf{x}}(\tau) \middle| \mathbf{x}(\tau)\right)_{L^2} \leq \\ - \left( A_{\mathbf{x}(\tau)+\mathrm{m}(\tau)} \frac{\partial \mathbf{x}(\tau)}{\partial x} \middle| \frac{\partial \mathbf{x}(\tau)}{\partial x} \right)_{L^2} + \\ |K(\tau)| \, \|\mathbf{x}(\tau)\|_{L^2} \, \|\mathbf{x}(\tau)\|_V + \\ \|\mathbf{x}(\tau)\|_{L^\infty} \left( |\Gamma_R(\tau)| + |\Gamma_L(\tau)| \right). \end{aligned}$$

Since, by (49)

$$\left( A_{\mathbf{x}(\tau)+\mathrm{m}(\tau)} \frac{\partial \mathbf{x}(\tau)}{\partial x} \middle| \frac{\partial \mathbf{x}(\tau)}{\partial x} \right)_{L^2} \geq \mu \, \|\mathbf{x}(\tau)\|_V^2 - \vartheta \, \|\mathbf{x}(\tau)\|_{L^2}^2$$

and

$$\|\mathbf{x}(\tau)\|_{L^\infty} \leq \sqrt{\frac{2\Lambda}{3}} \left\| \frac{\partial \mathbf{x}(\tau)}{\partial x} \right\|_{L^2},$$

for $\delta > 0$ we can estimate

$$\begin{aligned} \frac{1}{2} \frac{d}{d\tau} \|\mathbf{x}(\tau)\|_{L^2}^2 \leq \\ -\mu \, \|\mathbf{x}(\tau)\|_V^2 + \vartheta \, \|\mathbf{x}(\tau)\|_{L^2}^2 + \frac{1}{4\delta^2} K(\tau)^2 \, \|\mathbf{x}(\tau)\|_{L^2}^2 + \\ \delta^2 \, \|\mathbf{x}(\tau)\|_V^2 + \frac{1}{4\delta^2} \, |\Gamma(\tau)|^2 \, \frac{2\Lambda}{3} + \delta^2 \, \|\mathbf{x}(\tau)\|_V^2 = \\ \left(2\delta^2 - \mu\right) \|\mathbf{x}(\tau)\|_V^2 + \\ \left( \vartheta + \frac{1}{4\delta^2} K(\tau)^2 \right) \|\mathbf{x}(\tau)\|_{L^2}^2 + \frac{\Lambda}{6\delta^2} \, |\Gamma(\tau)|^2, \end{aligned} \tag{B.15}$$

where $|\Gamma(\tau)| = |\Gamma_R(\tau)| + |\Gamma_L(\tau)|$. Substituting $\delta = \sqrt{\frac{\mu}{2}}$ we get:

$$\frac{d}{d\tau}\|\mathbf{x}(\tau)\|^2_{L^2} \le \left(2\vartheta + \frac{1}{\mu}K(\tau)^2\right)\|\mathbf{x}(\tau)\|^2_{L^2} + \frac{2\Lambda}{3\mu}|\Gamma(\tau)|^2.$$

Thus for every $t \in \operatorname{dom}\mathbf{x}$ and $t < T$

$$\|\mathbf{x}(t)\|^2_{L^2} \le \left\|\overset{\circ}{w}\right\|^2_{L^2} + \int_0^t \left(2\vartheta + \frac{1}{\mu}K(\tau)^2\right)\|\mathbf{x}(\tau)\|^2_{L^2}\,d\tau + \frac{2\Lambda}{3\mu}\int_0^T |\Gamma(\tau)|^2\,d\tau.$$

From the Gronwall's inequality we get:

$$\|\mathbf{x}(t)\|^2_{L^2} \le \left(\left\|\overset{\circ}{w}\right\|^2_{L^2} + \frac{2\Lambda}{3\mu}\int_o^T |\Gamma(\tau)|^2\,d\tau\right) \exp\left(\int_0^t \left(2\vartheta + \frac{1}{\mu}K(\tau)^2\right)d\tau\right) \le C.$$

Upon substituting $\delta = \frac{\sqrt{\mu}}{2}$ in (B.15) it follows:

$$\frac{d}{d\tau}\|\mathbf{x}(\tau)\|^2_{L^2} \le -\mu\|\mathbf{x}(\tau)\|^2_V + \left(2\vartheta + \frac{2}{\mu}K(\tau)^2\right)\|\mathbf{x}(\tau)\|^2_{L^2} + \frac{4\Lambda}{3\mu}|\Gamma(\tau)|^2.$$

Consequently,

$$\|\mathbf{x}(\tau)\|^2_{L^2} \le \left\|\overset{\circ}{w}\right\|^2_{L^2} - \mu\int_0^t \|\mathbf{x}(\tau)\|^2_V\,d\tau + 2C\left(\vartheta T + \frac{1}{\mu}\int_0^T |K(\tau)|^2\,d\tau\right) + \frac{4\Lambda}{3\mu}\int_0^T |\Gamma(\tau)|^2\,d\tau$$

and finally

$$\int_0^t \|\mathbf{x}(\tau)\|^2_V\,d\tau \le \frac{1}{\mu}\left(\left\|\overset{\circ}{w}\right\|^2_{L^2} + 2C\left(\vartheta T + \frac{1}{\mu}\int_0^T |K(\tau)|^2\,d\tau\right)\right) + \frac{4\Lambda}{3\mu^2}\int_0^T |\Gamma(\tau)|^2\,d\tau.$$

It remains to proof that $\operatorname{dom}\mathbf{x} = [0, T)$.

Since $\mathbf{x}$ is a maximal solution of the equation (B.5) it is sufficient to show that $[0, T) \subset \operatorname{dom}\mathbf{x}$. We shall apply the theorem on extension of the solution of differential equations analogous to the equation (B.5) (Coddington et al., 1955). Accordingly, the proof will be complete if for any $0 \le T^* < T$ and a solution of equation (B.5) $\mathbf{y} : [0, T^*] \to X$ we have

$$\int_0^{T^*} \|\dot{\mathbf{y}}(\tau)\|_V\,d\tau < \infty.$$

Let $t \in \operatorname{dom}\dot{\mathbf{y}}$. Since $\mathbf{y}$ satisfies (B.5) we can estimate

$$\|\dot{\mathbf{y}}(t)\|_V \le \left\|B\left(A_{\mathbf{y}(t)+\mathrm{m}(t)}\frac{\partial \mathbf{y}(t)}{\partial x}\right)\right\|_V + |K(t)|\,\|B(\mathbf{y}(t))\|_V + \|G(\Gamma(t))\|_V.$$

By **Lemma A.2** and (B.9)

$$\|\dot{\mathbf{y}}(t)\|_V \le |B|\,\|A\|_{L^\infty}\|\mathbf{y}(t)\|_V + |K(t)|\,|B|\,C^{\frac{1}{2}} + |G|\,|\Gamma(t)|.$$

Since $\int_0^{T^*}|K(\tau)|\,d\tau < \infty$ and $\int_0^{T^*}|\Gamma(\tau)|\,d\tau < \infty$, thus we have obtained necessary condition. That completes the proof. □

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**INTERDIFFUSION IN R-COMPONENT (R ≥ 2) ONE DIMENSIONAL MIXTURE SHOWING CONSTANT CONCENTRATION**

Streszczenie

Pokazano, iż procesy dyfuzji w wieloskładnikowej mieszaninie prowadzą zawsze do jednoznacznie określonego stanu stacjonarnego. Do opisu procesu zastosowano postulat Darkena stwierdzający, iż strumień masy w r-składnikowej mieszaninie składa się z strumieni konwekcji i dyfuzji (proces określany jako jednowymiarowe zagadnienie dyfuzji wzajemnej). Prawa zachowania masy dla wszystkich składników, strumienie wyrażone zgodnie z postulatem Darkena, postulat stałości całkowitego stężenia w mieszaninie (spełniony, np. dla stopów metali) oraz warunki początkowe i brzegowe tworzą wewnętrznie spójne sformułowanie problemu dyfuzji wzajemnej. Wyprowadzono wariacyjną postać problemu i udowodniono: 1) istnienie słabego rozwiązania w układach otwartych i zamkniętych, 2) jednoznaczność takiego rozwiązania, 3) homogenizację mieszaniny dla długich czasów $t \to \infty$ oraz 3) zerowanie gradientów stężeń składników na brzegach układów zamkniętych.

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